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Proof of the Andrews–Dyson–Rhoades conjecture on the spt-crank

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ABSTRACT

The spt-crank of a vector partition, or an S-partition, was introduced by Andrews, Garvan and Liang. Let $N_S(m,n)$ denote the net number of S-partitions of n with spt-crank m, that is, the number of S-partitions (π_1,π_2,π_3) of n with spt-crank m such that the length of π_1 is odd minus the number of S-partitions (π_1,π_2,π_3) of n with spt-crank m such that the length of π_1 is even. Andrews, Dyson and Rhoades conjectured that $\{N_S(m,n)\}_m$ is unimodal for any n, and they showed that this conjecture is equivalent to an inequality between the rank and crank of ordinary partitions. They obtained an asymptotic formula for the difference between the rank and crank of ordinary partitions, which implies $N_S(m,n) \geq N_S(m+1,n)$ for sufficiently large n and fixed m. In this paper, we introduce a representation of an ordinary partition, called the m-Durfee rectangle symbol, which is a rectangular generalization of the Durfee symbol introduced by Andrews. We give a proof of the conjecture of Andrews, Dyson and Rhoades. We also show that this conjecture implies an inequality between the positive rank and crank moments obtained by Andrews, Chan and Kim.

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1. Introduction

In this paper, we give a proof of a conjecture of Andrews, Dyson and Rhoades on the spt-crank of a vector partition or an S-partition. The spt-function, called the smallest part function, was introduced by Andrews [2]. More precisely, we use $spt(n)$ to denote the total number of smallest parts in all partitions of n. For example, we have $spt(3) = 5$, $spt(4) = 10$ and $spt(5) = 14$. The smallest part function possesses many arithmetic properties analogous to the ordinary partition function, see, for example, [2,13,15,18].

Andrews [2] showed that the spt-function satisfies the following Ramanujan type congruences:

$$spt(5n + 4) \equiv 0 \pmod 5, \tag{1.1}$$

$$spt(7n + 5) \equiv 0 \pmod 7, \tag{1.2}$$

$$spt(13n + 6) \equiv 0 \pmod{13}. \tag{1.3}$$

To give combinatorial interpretations of the above congruences, Andrews, Garvan and Liang [6] introduced the spt-crank of an S-partition. Let $\mathcal{D}$ denote the set of partitions into distinct parts and $\mathcal{P}$ denote the set of partitions. For $\pi \in \mathcal{P}$, we use $s(\pi)$ to denote the smallest part of π with the convention that $s(\emptyset) = +\infty$. Let $\ell(\pi)$ denote the number of parts of π and $|\pi|$ denote the sum of parts of π. Define

$$S = \bigl\{(\pi_1, \pi_2, \pi_3) \in \mathcal{D} \times \mathcal{P} \times \mathcal{P}\colon\ \pi_1 \neq \emptyset \text{ and } s(\pi_1) \leq \min\bigl\{s(\pi_2), s(\pi_3)\bigr\}\bigr\}.$$

A triple (π_1, π_2, π_3) of partitions in S is called an S-partition, see Andrews, Garvan and Liang [6]. Moreover, if $|\pi_1| + |\pi_2| + |\pi_3| = n$, then (π_1, π_2, π_3) is called an S-partition of n. The spt-crank of an S-partition $\pi = (\pi_1, \pi_2, \pi_3)$, denoted $r(\pi)$, is defined to be the difference between the number of parts of π_2 and π_3, that is,

$$r(\pi) = \ell(\pi_2) - \ell(\pi_3).$$

For an S-partition $\pi = (\pi_1, \pi_2, \pi_3)$, we associate it with a sign $\omega(\pi) = (-1)^{\ell(\pi_1)-1}$ and let $|\pi|$ denote the sum of parts of π_1, π_2 and π_3, that is, $|\pi| = |\pi_1| + |\pi_2| + |\pi_3|$. Let $N_S(m, n)$ denote the net number of S-partitions of n with spt-crank m, that is,

$$N_S(m, n) = \sum_{\substack{|\pi|=n \\ r(\pi)=m}} \omega(\pi) \tag{1.4}$$

and

$$N_S(m, t, n) = \sum_{k \equiv m \pmod t} N_S(k, n).$$