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Decay of eigenfunctions of elliptic PDE's, I

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ABSTRACT

We study exponential decay of eigenfunctions of self-adjoint higher order elliptic operators on $\mathbb{R}^d$. We show that the possible (global) critical decay rates are determined algebraically. In addition we show absence of super-exponentially decaying eigenfunctions and a refined exponential upper bound.

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1. Introduction and results

Consider a real elliptic polynomial Q of degree q on $\mathbb{R}^d$. We consider the operator $H = Q(p)+V(x)$, $p = -\mathrm{i}\nabla$, on $L^2 = L^2(\mathbb{R}^d)$ with V real-valued, bounded and measurable and with $\lim_{|x|\to\infty} V(x) = 0$. By the assumptions on Q the operator $Q(p)$ is self-adjoint on the standard Sobolev space of order q which consequently is the domain of H too. The goal of the paper is to study exponential decay of L^2-eigenfunctions of H. It is the first in a series of two papers on the subject, the second one is [10].

We will mostly assume there is a splitting of V, $V = V_1 + V_2$, into real-valued bounded functions, V_1 smooth and V_2 measurable, with additional assumptions depending on the result. With virtually no complication of proofs V_2 could be taken complex-valued.

For a given $\lambda \in \mathbb{R}$ the energy surface

$$S_\lambda = \big\{(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \big| Q(\xi) = \lambda\big\}$$

is by definition regular if λ is not a critical value of Q, that is if

$$\nabla Q(\xi) \neq 0 \quad \text{on } S_\lambda. \tag{1.1}$$

We will need this condition in one of our results.

Suppose $(H-\lambda)\phi = 0$, $\phi \in L^2$. The *critical decay rate* is defined as

$$\sigma_{\mathrm{c}} = \sup\big\{\sigma \geq 0 \big| \mathrm{e}^{\sigma|x|}\phi \in L^2\big\}.$$

In this paper we shall study this notion of global decay rate for eigenfunctions, cf. previous works for the Laplacian [3,6–8] corresponding to the case $Q(\xi) = \xi^2$. We devote [10] to the study of the so-called local critical decay rate, see Subsection 1.3 for a definition and an announcement of some results of the second paper of this series. In particular we give in the present paper necessary (phase-space) conditions for a positive number σ to