Advances in Mathematics 270 (2015) 263–301

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Planar vortex patch problem in incompressible steady flow

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A R T I C L E   I N F O

Article history:
Received 27 January 2014
Accepted 7 September 2014
Available online 19 November 2014
Communicated by Ovidiu Savin

Keywords:
Steady solutions
Euler equation
Vortex patch
Variational method
Semilinear elliptic equations

A B S T R A C T

In this paper, we consider the planar vortex patch problem in an incompressible steady flow in a bounded domain Ω of $\mathbb{R}^2$. Let k be a positive integer and let κ_j be a positive constant, $j = 1, \ldots, k$. For any given non-degenerate critical point $\mathbf{x}_0 = (x_{0,1}, \ldots, x_{0,k})$ of the Kirchhoff–Routh function defined on Ω^k corresponding to $(\kappa_1, \ldots, \kappa_k)$, we prove the existence of a planar flow, such that the vorticity ω of this flow equals a large given positive constant λ in each small neighborhood of $x_{0,j}$, $j = 1, \ldots, k$, and $\omega = 0$ elsewhere. Moreover, as $\lambda \to +\infty$, the vorticity set $\{y\colon \omega(y) = \lambda\}$ shrinks to $\bigcup_{j=1}^{k}\{x_{0,j}\}$, and the local vorticity strength near each $x_{0,j}$ approaches κ_j, $j = 1, \ldots, k$.

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http://dx.doi.org/10.1016/j.aim.2014.09.027
0001-8708/

1. Introduction

The incompressible steady flow without external force is governed by the following mass equation

$$\nabla \cdot \mathbf{v} = 0, \tag{1.1}$$

and the following Euler motion equations

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P, \tag{1.2}$$

where $\mathbf{v}$ is the velocity and P is the pressure in the flow.

In a planar flow, the vorticity of the flow is defined by $\omega = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$. It follows from (1.1) that for an incompressible steady planar flow, in any simple connected domain Ω, there is a function ψ, which is called the stream function of the flow, such that

$$\mathbf{v} = \left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1}\right), \quad \text{in } \Omega. \tag{1.3}$$

Then the vorticity can be written as

$$\omega = \partial_1 v_2 - \partial_2 v_1 = -\Delta \psi. \tag{1.4}$$

The question on the existence of solutions representing steady vortex rings occupies a central place in the theory of vortex motion initiated by Helmholtz in 1858. In this paper, we will consider a steady planar flow of an ideal fluid in a bounded region and focus on the vortex patch problem. The planar vortex patch problem is to find a flow, such that the vorticity ω is a constant λ in a connected domain Ω_λ which shrinks to a single point as $\lambda \to +\infty$, while $\omega = 0$ elsewhere. This leads to the following free boundary problem

$$-\Delta \psi = \lambda 1_{\Omega_\lambda}, \tag{1.5}$$

where the region Ω_λ, which is called the vorticity set of the flow, is unknown. Here, we use 1_S to denote the characteristic function of a given set S. That is, $1_S = 1$ in S, and $1_S = 0$ elsewhere. Of course, it is also interesting to study the following generalized vortex patch problem: to find a flow, such that the vorticity ω is a constant λ in a region Ω_λ which has k connected components and shrinks to k different points $x_{0,j}$, $j = 1, \dots, k$, as $\lambda \to +\infty$, while $\omega = 0$ elsewhere. To solve this generalized vortex patch problem, we just need to study the existence of a solution for (1.5) such that Ω_λ has exactly k connected components.

Let Ω be a bounded simple connected domain in $\mathbb{R}^2$. In this paper, we consider the flow in Ω. Then the boundary condition is

$$\mathbf{v} \cdot \nu = 0, \quad \text{on } \partial\Omega, \tag{1.6}$$