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A new approach to pointwise heat kernel upper bounds on doubling metric measure spaces ☆

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ABSTRACT

On doubling metric measure spaces endowed with a strongly local regular Dirichlet form, we show some characterisations of pointwise upper bounds of the heat kernel in terms of global scale-invariant inequalities that correspond respectively to the Nash inequality and to a Gagliardo–Nirenberg type inequality when the volume growth is polynomial. This yields a new proof and a generalisation of the well-known equivalence between classical heat kernel upper bounds and relative Faber–Krahn inequalities or localised Sobolev or Nash inequalities. We are able to treat more general pointwise estimates, where the heat kernel rate of decay is not necessarily governed by the volume growth. A crucial role is played by the finite propagation speed property for the associated wave equation, and our main result holds for an abstract semigroup of operators satisfying the Davies–Gaffney estimates.

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1. Introduction

1.1. Background and motivation

Let M be a complete non-compact connected Riemannian manifold, d be the geodesic distance and μ be the Riemannian measure on M. Denote by $V(x,r) := \mu(B(x,r))$ the volume of the ball $B(x,r)$ of centre $x \in M$ and radius $r > 0$ with respect to d.

Let Δ be the non-negative Laplace–Beltrami operator and p_t be the heat kernel on M, that is by definition the smallest positive fundamental solution of the heat equation:

$$\frac{\partial u}{\partial t} + \Delta u = 0,$$

or the kernel of the heat semigroup $e^{-t\Delta}$, i.e.

$$e^{-t\Delta} f(x) = \int_M p_t(x,y) f(y) d\mu(y), \quad f \in L^2(M,\mu),\ \mu\text{-a.e. } x \in M.$$

It is well-known that in this situation, contrary to more general ones, $p_t(x,y)$ is smooth in $t > 0$, $x, y \in M$ and everywhere positive (see for instance [37]).

In the Euclidean space $\mathbb{R}^n$, p_t is given by the classical Gauss–Weierstrass kernel:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right), \quad t > 0,\ x, y \in \mathbb{R}^n.$$