Advances in Mathematics 270 (2015) 400–456

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Existence of pulses in excitable media with nonlocal coupling ☆

Grégory Faye, Arnd Scheel *

*University of Minnesota, School of Mathematics, 206 Church Street S.E., Minneapolis, MN 55455, USA*

ARTICLE INFO

*Article history:*
Received 26 November 2013
Accepted 3 November 2014
Available online 20 November 2014
Communicated by Charles Fefferman

*Keywords:*
Traveling wave
Nonlocal equation
FitzHugh–Nagumo system
Fredholm operators

ABSTRACT

We prove the existence of fast traveling pulse solutions in excitable media with non-local coupling. Existence results had been known, until now, in the case of local, diffusive coupling and in the case of a discrete medium, with finite-range, non-local coupling. Our approach replaces methods from geometric singular perturbation theory, that had been crucial in previous existence proofs, by a PDE oriented approach, relying on exponential weights, Fredholm theory, and commutator estimates.



## 1. Introduction

Excitable media play a central role in our understanding of complex systems. Chemical reactions [2,19], calcium waves [34], and neural field models [6,7] are among the examples that motivate our present study. A prototypical model of excitable kinetics are

☆ G.F. was partially supported by the National Science Foundation through grant NSF-DMS-1311414. A.S. was partially supported by the National Science Foundation through grants NSF-DMS-0806614 and NSF-DMS-1311740.

* Corresponding author.
*E-mail address:* scheel@umn.edu (A. Scheel).

http://dx.doi.org/10.1016/j.aim.2014.11.005

the FitzHugh–Nagumo kinetics, derived first as a simplification of the Hodgkin–Huxley model for the propagation of electric signals through nerve fibers [20],

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u) - v, \tag{1.1a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \epsilon(u - \gamma v), \tag{1.1b}$$

where, for instance, $f(u) = u(1-u)(u-a)$. For $0 < a < 1/2$ and $\gamma > 0$, not too large, all trajectories in this system converge to the trivial equilibrium $u = v = 0$. The system is however excitable in the sense that finite-size perturbations of $u$, past the excitability threshold $a$, away from the stable equilibrium $u = v = 0$, can induce a long transient, where $f(u) \sim v$, $u > 1/2$. During these transients, which last for times $\mathcal{O}(1/\epsilon)$, $u$ is said to be in the excited state; eventually, $u$ returns to values $f(u) \sim v$, $u < 1/2$, the quiescent state.

Interest in these systems stems from the fact that, although kinetics are very simple and ubiquitous in nature, with convergence of all trajectories to a simple stable equilibrium, spatial coupling can induce quite complex dynamics. The simplest example is the propagation of a stable excitation pulse, more complicated examples include two-dimensional spiral waves and spatio-temporal chaos. Intuitively, a local excitation can trigger excitations of neighbors before decaying back to the quiescent state in a spatially coupled system. After initial transients, one then observes a spatially propagating region where $u$ belongs to the excited state.

Rigorous approaches to the existence of such excitation pulses have been based on singular perturbation methods. Consider, for example,

$$\partial_t \mathbf{u}(x,t) = \partial_{xx}\mathbf{u}(x,t) + f\big(\mathbf{u}(x,t)\big) - \mathbf{v}(x,t), \tag{1.2a}$$

$$\partial_t \mathbf{v}(x,t) = \epsilon\big(\mathbf{u}(x,t) - \gamma\mathbf{v}(x,t)\big) \tag{1.2b}$$

with $x \in \mathbb{R}$. One looks for solutions of the form

$$(\mathbf{u},\mathbf{v})(x,t) = (\mathbf{u},\mathbf{v})(x-ct), \tag{1.3}$$

and finds first-order ordinary differential equation for $\mathbf{u}, \mathbf{u}_x, \mathbf{v}$, in which one looks for a homoclinic solution to the origin. The small parameter $\epsilon$ introduces a singularly perturbed structure into the problem which allows one to find such a homoclinic orbit by tracking stable and unstable manifolds along fast intersections and slow, normally hyperbolic manifolds [9,18,24]. This approach has been successfully applied in many other contexts with slow-fast like structures, with higher- or even infinite-dimensional slow-fast ODEs; see for instance [22,23,36].

Our interest is in media with *infinite-range* coupling. We will focus on *linear* coupling through convolutions, although we believe that the existence result extends to a variety of other problems. To fix ideas, we consider