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Explicit Euclidean embeddings in permutation invariant normed spaces

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ABSTRACT

Let $(X, \|\cdot\|)$ be a real normed space of dimension $N \in \mathbb{N}$ with a basis $(e_i)_1^N$ such that the norm is invariant under coordinate permutations. Assume for simplicity that the basis constant is at most 2. Consider any $n \in \mathbb{N}$ and $0 < \varepsilon < 1/4$ such that $n \leq c(\log \varepsilon^{-1})^{-1} \log N$. We provide an explicit construction of a matrix that generates a $(1+\varepsilon)$ embedding of ℓ_2^n into X.

1. Introduction

The modern formulation of Dvoretzky's theorem states that there exists a function $\xi(\cdot) : (0, 1/2) \to (0, \infty)$ such that for all $(N, n, \varepsilon) \in \mathbb{N} \times \mathbb{N} \times (0, 1/2)$ with $n \leq \xi(\varepsilon) \log N$, and any real normed space $(X, \|\cdot\|)$ with $\dim(X) = N$, there exists a linear map $T : \mathbb{R}^n \to X$ such that for all $x \in \mathbb{R}^n$,

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$$(1-\varepsilon)|x| \le \|Tx\| \le (1+\varepsilon)|x| \tag{1}$$

where $|\cdot|$ denotes the standard Euclidean norm. The normed space $(\mathbb{R}^n, |\cdot|)$ is denoted ℓ_2^n. Eq. (1) expresses the fact that ℓ_2^n can be $(1+\varepsilon)$ embedded in X. Geometrically, this means that any centrally symmetric convex body in high dimensional Euclidean space has cross-sections of lower dimension that are approximately ellipsoidal. The logarithmic dependence on N, which is due to Milman [16], is optimal in the general setting (specifically for $X = \ell_\infty^N$) but can be greatly improved for other spaces such as ℓ_1^N, where one can take $n = \lfloor c\varepsilon^2 N \rfloor$ for some universal constant $c > 0$. On the other hand, the optimal dependence on ε is unknown. The best current bound is $\xi(\varepsilon) = c\varepsilon(\log \varepsilon^{-1})^{-2}$ by Schechtman [20]. If weaker forms of Knaster's problem are true [18,12], then one could take $\xi(\varepsilon) = c(\log \varepsilon^{-1})^{-1}$, which would be optimal since this is the correct dependence in ℓ_∞^N. We refer to [17,20,21] for a more detailed background.

The proofs of Dvoretzky's theorem typically make use of random embeddings and are nonconstructive. A natural question (see for example Section 2.2 in [11] and Section 4 in [22]) is whether or not one can make these and other probabilistic constructions in functional analysis explicit, or at least decrease the randomness in some way.

Of course it is impossible to find an explicit Euclidean subspace of a completely general and unspecified normed space, otherwise all subspaces would be Euclidean. This follows from rotational invariance of the class of symmetric convex bodies in $\mathbb{R}^N$ and the fact that the orthogonal group $O(N)$ acts transitively on the Grassmannian $G_{N,n}$. The same is true even if we assume that the space has nontrivial cotype, or that the unit ball is in John's position. One either has to construct explicit subspaces for specific spaces individually, or (most likely) we need to impose some sort of symmetry in order to get a grip on the space.

The case $X = \ell_1^N$ is particularly important from the point of view of applications, see for example [10] and the references therein. For this space, there are various algorithms to compute embeddings or decrease randomness [1,2,5,6,8–10,15,19], although there is still no truly explicit embedding that is as good as a random one. For $X = \ell_p^N$, where $p \in \mathbb{N}$ is an even integer and

$$N \ge \binom{n+p-1}{p}$$

which is true whenever $n \le N^{1/p}$, the space ℓ_2^n embeds isometrically into X, [18,13]. In this case there are also various explicit embeddings. For example, the identity

$$6\left(\sum_{i=1}^{4} x_i^2\right)^2 = \sum_{1 \le i < j \le 4} (x_i + x_j)^4 + \sum_{1 \le i < j \le 4} (x_i - x_j)^4$$

defines an isometric embedding of ℓ_2^4 into ℓ_4^{12}. As far as we are aware, there are no known explicit embeddings (in the classical sense) or even algorithms, that apply to a