Advances in Mathematics 269 (2015) 1–55

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

There are only finitely many distance-regular graphs of fixed valency greater than two

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ARTICLE INFO

Article history:
Received 16 September 2014
Accepted 22 September 2014
Available online 22 October 2014
Communicated by Charles Fefferman

Keywords:
Distance-regular graphs
Bannai–Ito conjecture
Sims conjecture
Othogonal polynomials
Christoffel numbers

ABSTRACT

In this paper we prove the Bannai–Ito conjecture, namely that there are only finitely many distance-regular graphs of fixed valency greater than two.

Contents

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http://dx.doi.org/10.1016/j.aim.2014.09.025
0001-8708/

1. Introduction

A finite, connected graph Γ with vertex set $V(\Gamma)$ and path-length distance d is said to be *distance-regular* if, for any vertices $x, y \in V(\Gamma)$ and any integers $1 \leq i, j \leq \max\{d(z,w) : z, w \in V(\Gamma)\}$, the number of vertices at distance i from x and distance j from y depends only on i, j and $d(x,y)$, independent of the choice of x and y. Many distance-regular graphs arise from classical objects, such as the Hamming graphs, the Johnson graphs, the Grassmann graphs, the bilinear forms graphs, and the dual polar graphs amongst others. In particular, distance-regular graphs give a framework to study these classical objects from a combinatorial point of view. In addition, distance-regular graphs and association schemes give an algebraic–combinatorial framework to study, for example, codes and designs [13,18].

In their 1984 book, E. Bannai and T. Ito conjectured that there are only finitely many distance-regular graphs of fixed valency greater than two (cf. [5, p. 237]). In this paper we prove that their conjecture holds:

Theorem 1.1. *There are only finitely many distance-regular graphs of fixed valency greater than two.*

History

A distance-transitive graph is a connected graph Γ such that for every four (not necessarily distinct) vertices x, y, u, v in $V(\Gamma)$ with $d(x,y) = d(u,v)$, there exists an automorphism τ of Γ such that $\tau(x) = u$ and $\tau(y) = v$ both hold. It is straight-forward to see that distance-transitive graphs are distance-regular graphs. In [14,15], P.J. Cameron, C.E. Praeger, J. Saxl and G.M. Seitz proved that there are only finitely many finite distance-transitive graphs of fixed valency greater than two. They did this by applying Sims' conjecture [33] for finite permutation groups (i.e. that there exists an integral function f such that $|G_x| \leq f(d_{G_x})$ holds, where, for G a primitive permutation group