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Global existence for some transport equations with nonlocal velocity

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A R T I C L E   I N F O

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This paper is dedicated to Professor Eitan Tadmor on the occasion of his 60th birthday

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A B S T R A C T

In this paper, we study transport equations with nonlocal velocity fields with rough initial data. We address the global existence of weak solutions of a one dimensional model of the surface quasi-geostrophic equation and the incompressible porous media equation, and one dimensional and n dimensional models of the dissipative quasi-geostrophic equations and the dissipative incompressible porous media equation.

1. Introduction

In this paper, we study several active scalar equations with nonlocal velocity fields. Here, the non-locality means that the velocity field is defined through a singular integral operator that is represented in terms of a Fourier multiplier. For example, in the

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two dimensional Euler equation in vorticity form [6], the velocity is recovered from the vorticity ω through

$$u = \nabla^\perp(-\Delta)^{-1}\omega \quad \text{or equivalently} \quad \widehat{u}(\xi) = \frac{i\xi^\perp}{|\xi|^2}\widehat{\omega}(\xi). \tag{1.1}$$

Other nonlocal and quadratically nonlinear equations appear in many applications. Prototypical examples are the surface quasi-geostrophic equation, the incompressible porous medium equation, Stokes equations, magnetogeostrophic equation and their variants. We briefly introduce the equations below.

The surface quasi-geostrophic equations. The surface quasi-geostrophic equation describes the dynamics of the mixture of cold and hot air and the fronts between them in 2 dimensions [23,51]. The equation is of the form

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \tag{1.2}$$

where the scalar function θ is the potential temperature and $\mathcal{R}_j$ is the Riesz transform

$$\mathcal{R}_j f(x) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{R}^2} \frac{(x_j - y_j)f(y)}{|x-y|^3}dy, \quad j = 1, 2.$$

Similar model equations of (1.2) with different types of nonlocal velocities are proposed and analyzed in [2,9,19], respectively (see also [4,10,18,40,44,42,43]):

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \mathcal{R}\theta,$$
$$\theta_t + \nabla\cdot(\theta\mathcal{R}\theta) = 0,$$
$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \nabla^\perp\Lambda^{\beta-2}\theta,\ 1<\beta\le 2.$$

We finally introduce the two dimensional Euler-α model in vorticity form:

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \nabla^\perp\Lambda^{-2+\alpha}\theta,\ \alpha\in[0,1] \tag{1.3}$$

which interpolates between (1.1) ($\alpha = 0$) and (1.2) ($\alpha = 1$).

The incompressible porous medium equation. This equation takes the form [17]

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \mathcal{R}^\perp\mathcal{R}_1\theta, \tag{1.4}$$

where θ is now the density of the incompressible fluid moving through a homogeneous porous medium. The following version also has been studied with $\beta > 0$ [31]:

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \Lambda^\beta\mathcal{R}^\perp\mathcal{R}_1\theta. \tag{1.5}$$