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[www.elsevier.com/locate/aim](http://www.elsevier.com/locate/aim)Parking spaces <sup>☆</sup>Drew Armstrong<sup>a</sup>, Victor Reiner<sup>b,\*</sup>, Brendon Rhoades<sup>c</sup><sup>a</sup> Dept. of Mathematics, University of Miami, Coral Gables, FL 33146, United States<sup>b</sup> School of Mathematics, University of Minnesota, Minneapolis, MN 55455, United States<sup>c</sup> Dept. of Mathematics, University of California, San Diego, La Jolla, CA 92093, United States

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## ABSTRACT

Let  $W$  be a Weyl group with root lattice  $Q$  and Coxeter number  $h$ . The elements of the finite torus  $Q/(h+1)Q$  are called the  $W$ -parking functions, and we call the permutation representation of  $W$  on the set of  $W$ -parking functions the (standard)  $W$ -parking space. Parking spaces have interesting connections to enumerative combinatorics, diagonal harmonics, and rational Cherednik algebras. In this paper we define two new  $W$ -parking spaces, called the noncrossing parking space and the algebraic parking space, with the following features:

- They are defined more generally for real reflection groups.
- They carry not just  $W$ -actions, but  $W \times C$ -actions, where  $C$  is the cyclic subgroup of  $W$  generated by a Coxeter element.
- In the crystallographic case, both are isomorphic to the standard  $W$ -parking space.

Our Main Conjecture is that the two new parking spaces are isomorphic to each other as permutation representations of  $W \times C$ . This conjecture ties together several threads in the Catalan combinatorics of finite reflection groups. Even the

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weakest form of the Main Conjecture has interesting combinatorial consequences, and this weak form is proven in all types except  $E_7$  and  $E_8$ . We provide evidence for the stronger forms of the conjecture, including proofs in some cases, and suggest further directions for the theory.

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1. Introduction

Let  $W$  be a finite Coxeter group (finite real reflection group). (We refer to the standard references [10,30].) The main goal of this paper is to define two new objects that deserve

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