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Splitting families of sets in ZFC

Menachem Kojman [1]

Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er Sheva 84105, Israel

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Miller's 1937 splitting theorem was proved for every finite $n > 0$ for all ρ-uniform families of sets in which ρ is infinite. A simple method for proving Miller-type splitting theorems is presented here and an extension of Miller's theorem is proved in ZFC for every cardinal ν for all ρ-uniform families in which $\rho \geq \beth_\omega(\nu)$. The main ingredient in the method is an asymptotic infinitary Löwenheim–Skolem theorem for anti-monotone set functions.

As corollaries, the use of additional axioms is eliminated from splitting theorems due to Erdős and Hajnal [1], Komjáth [7], Hajnal, Juhász and Shelah [4]; upper bounds are set on conflict-free coloring numbers of families of sets; and a general comparison theorem for ρ-uniform families of sets is proved, which generalizes Komjáth's comparison theorem for $\aleph_0$-uniform families [8].

E-mail address: kojman@math.bgu.ac.il.

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1. Introduction

All modern theorems on splitting families of infinite sets follow Miller's seminal 1937 inductive proof [10], in which Miller assumes that an arbitrary ρ-uniform family of sets satisfies that the intersection of any ρ^+ of its members has cardinality smaller than some fixed *finite* number n.

Erdős and Hajnal devoted [1] to relaxing Miller's condition, which they denoted by $C(\rho^+, n)$, to $C(\rho^+, \nu)$ with a given infinite ν for all sufficiently large ρ. With the Generalized Continuum Hypothesis adopted as an additional axiom, they generalized Miller's theorem to all infinite ν and ρ such that $\rho > \nu^+$.

Hajnal, Juhász and Shelah [4] proved a sophisticated general theorem, with a rather difficult proof, that enabled the relaxation of the GCH axiom to a considerably weaker additional axiom, some weak variant of the Singular Cardinals Hypothesis, in proving several Miller-type theorems, including the one by Erdős and Hajnal. These results were proved for an infinite ν and $\rho > 2^\nu$.

Here we prove those and a few other Miller type theorems in ZFC with no additional axioms for an arbitrary ν and $\rho \geq \beth_\omega(\nu)$. We present and use a general method, which extends the method that was used in [6] for infinite graphs colorings. This new method does not require any specialized notions and is not more complicated than Miller's original method. It may be useful in other contexts as well. Let us describe next the set-theoretic development that made this method possible.

Miller's original proof used a closure argument under finitary operations, which amounts to a weak version of the well-known Löwenheim–Skolem theorem, to obtain filtrations of arbitrary large families, from which the inductive argument could be carried out. With $C(\rho^+, \nu)$ replacing $C(\rho^+, n)$, the closure needs to be formed with respect to an infinitary operation. However, there is no general Löwenheim–Skolem theorem for ν-ary operations, for a good reason: the equation $\lambda^n = \lambda$, which entails the Löwenheim–Skolem theorem at λ, holds for every infinite λ and finite $n > 0$, but $\lambda^\nu = \lambda$, which implies the Löwenheim–Skolem theorem at λ for ν-ary operations, fails periodically by König's lemma and, worse still, cannot be generally upper-bounded in terms of λ on the basis of ZFC.

Erdős and Hajnal used in [1] a GCH theorem by Tarski [15] to obtain filtrations with respect to the required closure notion, as did Komjáth in [7]. Hajnal, Juhász and Shelah [4] used a weak version of the SCH and L-like combinatorial principles in their proof, in which filtrations are replaced by a more sophisticated abstract notion. Here we use simple arithmetic relations concerning the density of κ-subsets of λ to prove an asymptotic Löwenheim–Skolem theorem and obtain filtrations in ZFC for order-reversing set operations. The theorem holds at all sufficiently large λ.

What makes this possible is Shelah's spectacular revised GCH theorem [12] (which, of course, was not available at the time [4] was published). This theorem is Shelah's most important achievement in pcf theory. It provides, for the first time since