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On the dimension of the graph of the classical Weierstrass function ☆

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A B S T R A C T

This paper examines dimension of the graph of the famous Weierstrass non-differentiable function

$$W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$$

for an integer $b \geq 2$ and $1/b < \lambda < 1$. We prove that for every b there exists (explicitly given) $\lambda_b \in (1/b, 1)$ such that the Hausdorff dimension of the graph of $W_{\lambda,b}$ is equal to $D = 2 + \frac{\log \lambda}{\log b}$ for every $\lambda \in (\lambda_b, 1)$. We also show that the dimension is equal to D for almost every λ on some larger interval. This partially solves a well-known thirty-year-old conjecture. Furthermore, we prove that the Hausdorff dimension of the graph of the function

$$f(x) = \sum_{n=0}^{\infty} \lambda^n \phi(b^n x)$$

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for an integer $b \geq 2$ and $1/b < \lambda < 1$ is equal to D for a typical $\mathbb{Z}$-periodic C^3 function ϕ.

1. Introduction and statements

This paper is devoted to the study of dimension of the graphs of functions of the form

$$f^{\phi}_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \phi\big(b^n x\big) \tag{1.1}$$

for $x \in \mathbb{R}$, where $b > 1$, $1/b < \lambda < 1$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a non-constant $\mathbb{Z}$-periodic Lipschitz continuous piecewise C^1 function. The famous example

$$W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\big(2\pi b^n x\big),$$

for $\phi(x) = \cos(2\pi x)$, was introduced by Weierstrass in 1872 as one of the first examples of a continuous nowhere differentiable function on the real line. In fact, Weierstrass proved the non-differentiability for some values of the parameters, while the complete proof was given by Hardy [9] in 1916. Later, starting from the work of Besicovitch and Ursell [5], the graphs of $f^{\phi}_{\lambda,b}$ and related functions were studied from a geometric point of view as fractal curves in the plane.

The graph of a function $f^{\phi}_{\lambda,b}$ of the form (1.1) is approximately self-affine with scales λ and $1/b$, which suggests that its dimension should be equal to

$$D = 2 + \frac{\log \lambda}{\log b}.$$

Indeed, Kaplan, Mallet-Paret and Yorke [12] proved that the box dimension of the graph of every function (1.1) or, more general, every function of the form

$$f(x) = \sum_{n=0}^{\infty} \lambda^n \phi\big(b^n x + \theta_n\big), \tag{1.2}$$

where $\theta_n \in \mathbb{R}$, is equal to D. However, the question of determining the Hausdorff dimension turned out to be much more complicated.

In 1986, Mauldin and Williams [18] proved that if a function f has the form (1.2), then for given D there exists a constant $C > 0$ such that the Hausdorff dimension of the graph is larger than $D - C/\log b$ for large b. Shortly after, Przytycki and Urbański showed in [21] that the Hausdorff dimension of the graph of f is larger than 1 under some weaker assumptions.