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Stable motivic π_1 of low-dimensional fields

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ABSTRACT

Let k be a field with cohomological dimension less than 3; we call such fields *low-dimensional*. Examples include algebraically closed fields, finite fields and function fields thereof, local fields, and number fields with no real embeddings. We determine the 1-column of the motivic Adams–Novikov spectral sequence over k. Combined with rational information we use this to compute $\pi_1\mathsf{S}$, the first stable motivic homotopy group of the sphere spectrum over k. Our main result affirms Morel's π_1-conjecture in the case of low-dimensional fields. We also determine $\pi_{1+n\alpha}\mathsf{S}$ for weights $n \in \mathbb{Z} \setminus \{-2, -3, -4\}$.

1. Introduction

The stable motivic homotopy groups of the sphere spectrum over a field form an interesting and computationally challenging class of invariants. They are at least as difficult to compute as the stable homotopy groups of the topological sphere spectrum (cf. [19] for a precise sense in which this is true), but also incorporate a great deal of

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nuanced arithmetic information about the base field. These groups were first explored by Morel [27,28], who computed the 0-th group as the Grothendieck–Witt ring of quadratic forms. Work of Dugger and Isaksen [11] and Hu, Kriz and Ormsby [17] produced hands-on computations in a range of dimensions over algebraically closed fields, but results encompassing a richer class of fields remained elusive.

Given a field k, let p denote the exponential characteristic of k ($p = 1$ if $\operatorname{char} k = 0$; otherwise $p = \operatorname{char} k$), and let $\mathsf{S}[1/p]$ denote the motivic sphere spectrum over k with p inverted,

$$\mathsf{S}[1/p] = \operatorname{hocolim}(\mathsf{S} \xrightarrow{p} \mathsf{S} \xrightarrow{p} \cdots).$$

In particular, if $p = 1$ then $\mathsf{S}[1/p] = \mathsf{S}$. In this paper we determine the first stable motivic homotopy group of $\mathsf{S}[1/p]$, $\pi_1\mathsf{S}[1/p]$, over any field k with cohomological dimension less than 3. In fact, when k satisfies these hypotheses and $p \neq 2, 3$, we prove a variant of Morel's conjecture on $\pi_1\mathsf{S}$ stating that there is a short exact sequence

$$0 \to K_2^M(k)/24 \to \pi_1\mathsf{S}[1/p] \to K_1^M(k)/2 \oplus \mathbb{Z}/2 \to 0.$$

Here, $K_*^M(k)$ denotes Milnor K-theory, defined by generators and relations in [25]. The short exact sequence for $\pi_1\mathsf{S}[1/p]$ is a stable version of [1, Conjecture 7].

Our cohomological dimension assumption holds for many examples of interest, e.g., algebraically closed fields, finite fields, local fields, and number fields with no real embeddings [33]. The factor of 24 is related to the fact that π_3 of the topological sphere spectrum is cyclic of that order, and also to the computation of unstable $\pi_3(\mathbb{A}^3 \smallsetminus 0)$ by Asok and Fasel in their work on splittings of vector bundles [3].

To put our result in context, recall that Morel and Voevodsky [31] constructed a homotopy theory of smooth k-schemes in which the affine line, $\mathbb{A}^1$, plays the rôle of the unit interval; this is called motivic homotopy theory. Whenever we have a homotopy theory of a particular type of spaces, we may invert the smash product with a space T by forming a category of T-spectra. In motivic homotopy theory, inverting $T = \mathbb{P}^1$ corresponds to inverting the Lefschetz motive, and the resulting homotopy theory of $\mathbb{P}^1$-spectra is called stable motivic homotopy theory. There are several Quillen equivalent model category structures on $\mathbb{P}^1$-spectra giving the same stable homotopy theory. For concreteness and convenience, we will work with Bousfield–Friedlander $\mathbb{P}^1$-spectra as in [18, Theorem 2.9].

In order to detect weak equivalences of $\mathbb{P}^1$-spectra, one applies the *homotopy presheaf functor*, $\underline{\pi}_\star$. Here $\star$ stands for all indices of the form $(m, n) \in \mathbb{Z}^2$. Following a grading convention inspired by $\mathbb{Z}/2$-equivariant homotopy theory, we write $m + n\alpha$ for (m, n), and let $S^{m+n\alpha} = (S^1)^{\wedge m} \wedge (\mathbb{G}_m)^{\wedge n}$. With this definition we can identify $\mathbb{P}^1$ with the $(1 + \alpha)$-sphere in the motivic homotopy category. Then for a $\mathbb{P}^1$-spectrum E, $\underline{\pi}_{m+n\alpha}(\mathsf{E})$ is the presheaf taking a smooth k-scheme X to

$$(\underline{\pi}_{m+n\alpha}\mathsf{E})(X) = \left[S^{m+n\alpha} \wedge X_+, \mathsf{E}\right],$$