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On the dual nature of partial theta functions and Appell–Lerch sums

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A B S T R A C T

In recent work, Hickerson and the author demonstrated that it is useful to think of Appell–Lerch sums as partial theta functions. This notion can be used to relate identities involving partial theta functions with identities involving Appell–Lerch sums. In this sense, Appell–Lerch sums and partial theta functions appear to be dual to each other. This duality theory is not unlike that found by Andrews between various sets of identities of Rogers–Ramanujan type with respect to Baxter's solution to the hard hexagon model of statistical mechanics. As an application we construct bilateral q-series with mixed mock modular behaviour. In subsequent work we see that our bilateral series are well-suited for computing radial limits of Ramanujan's mock theta functions.

0. Notation

Let q be a nonzero complex number with $|q| < 1$ and define $\mathbb{C}^* := \mathbb{C} - \{0\}$. Recall

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$$(x)_n = (x;q)_n := \prod_{i=0}^{n-1}(1-q^ix), \qquad (x)_\infty = (x;q)_\infty := \prod_{i\geq 0}(1-q^ix), \quad \text{and}$$

$$j(x;q) := (x)_\infty(q/x)_\infty(q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}}x^n, \tag{0.1}$$

where in the last line the equivalence of product and sum follows from Jacobi's triple product identity. Here a and m are integers with m positive. Define

$$J_{a,m} := j(q^a;q^m), \qquad J_m := J_{m,3m} = \prod_{i\geq 1}(1-q^{mi}), \quad \text{and} \quad \overline{J}_{a,m} := j(-q^a;q^m).$$

We will use the following definition of an Appell–Lerch sum [19]:

$$m(x,q,z) := \frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}z^r}{1-q^{r-1}xz}. \tag{0.2}$$

The symbol $\sum^*$ indicates convergence problems, so care should be taken.

1. Introduction

In his last letter to Hardy, Ramanujan gave a list of seventeen functions which he called "mock theta functions". [25, p. xxxi]: "*I am extremely sorry for not writing you a single letter up to now... I discovered very interesting functions recently which I call 'Mock' ϑ-functions. Unlike the 'False' ϑ-theta functions (studied partially by Prof. Rogers in his interesting paper) they enter mathematics as beautifully as the ordinary theta functions...*". Each mock theta function was defined by Ramanujan as a q-series convergent for $|q| < 1$. He stated that they have certain asymptotic properties as q approaches a root of unity, similar to the properties of ordinary theta functions, but that they are not theta functions. He also stated several identities relating some of the mock theta functions to each other. Later, many more mock theta function identities were found in the Lost Notebook [26].

Numerous entries in the Lost Notebook expand Eulerian forms (q-hypergeometric series) in terms of theta functions (Rogers–Ramanujan type identities), Appell–Lerch sums (mock theta functions), or partial theta functions. Although partial theta functions are arguably the least understood ones, they do play significant roles in areas outside of number theory such as quantum invariants of 3-manifolds [20]. Appell–Lerch sums also appear naturally in the context of black hole physics [15]. One wants to understand the various types of Eulerian forms and how they relate to each other. In this direction, Andrews [5] has recently produced q-hypergeometric formulas which simultaneously prove mock theta function identities and Rogers–Ramanujan type identities.