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The Cauchy integral in $\mathbb{C}^n$ for domains with minimal smoothness

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We prove $L^p(b\mathcal{D})$-regularity of the Cauchy–Leray integral for bounded domains $\mathcal{D} \subset \mathbb{C}^n$ whose boundary satisfies the minimal regularity condition of class $C^{1,1}$, together with a naturally occurring notion of convexity.

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1. Introduction

The purpose of this paper is to study the Cauchy integral in several complex variables and in particular to establish its L^2 (and L^p) boundedness in the setting of minimal smoothness assumptions on the boundary of the domain. We take as our model the well-known one-dimensional theory of Calderón [9], Coifman–McIntosh–Meyer [13] and David [14], and its key theorem concerning the boundedness in L^p of the Cauchy integral for a Lipschitz domain. Our goal is to find an extension to $\mathbb{C}^n$ of these results, and in doing so we see that the context $n > 1$ requires that we recast the problem to take into account its geometric setting, and also overcome inherent difficulties that do not arise in the case $n = 1$. To describe our results we begin by sketching the background.

1.1. Situation for $n = 1$

The initial result was the classical theorem of M. Riesz for the Cauchy integral of the disc (i.e. the Hilbert transform on the circle) which gave the boundedness on L^p, for $1 < p < \infty$. The standard proofs which developed for this then allowed an extension to a corresponding result where the disc is replaced by a domain $\mathcal{D}$ whose boundary is relatively smooth, i.e. of class $C^{1+\epsilon}$, for $\epsilon > 0$. However going beyond that to the limiting case of regularity, namely C^1 and other variants "near" C^1, required further ideas. Incidentally, the techniques introduced in this connection led to significant developments in harmonic analysis such as the "$T(1)$ theorem", and various aspects of multilinear analysis and analytic capacity see e.g., [11] and [12], [33,36–39] and [45]. The importance of those advances suggests the natural question: *what might be the corresponding results for the Cauchy integral in several variables*?

1.2. Problem for $n > 1$

When we turn to higher dimensions we see at once two basic differences which are present, that have no analogue in one dimension.

• The role of pseudo-convexity. That the pseudo-convexity of the underlying domain is a prerequisite can be understood from a variety of points of view, one of which is discussed below. For us the key consequence of this is that this condition, which essentially involves two degrees of differentiability of the boundary, implies that the correct limiting condition of smoothness should be "near" C^2, as opposed to near C^1 in one dimension.

• The fact that given a domain $\mathcal{D}$ there is an infinitude of different "Cauchy integrals" that present themselves, as opposed to when $n = 1$. This raises the further problem of finding (or constructing) the Cauchy integral appropriate for each domain that will be considered. The starting point for such constructions is the Cauchy–Fantappiè formalism,