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On instability and stability of three-dimensional gravity driven viscous flows in a bounded domain

Fei Jiang[a,b,*], Song Jiang[b]

[a] College of Mathematics and Computer Science, Fuzhou University, Fuzhou, 350108, China
[b] Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China

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ABSTRACT

We investigate the instability and stability of some steady-states of a three-dimensional nonhomogeneous incompressible viscous flow driven by gravity in a bounded domain Ω of class C^2. When the steady density is heavier with increasing height (i.e., the Rayleigh–Taylor steady-state), we show that the steady-state is linear unstable (i.e., the linear solution grows in time in H^2) by constructing a (standard) energy functional and exploiting the modified variational method. Then, by introducing a new energy functional and using a careful bootstrap argument, we further show that the steady-state is nonlinear unstable in the sense of Hadamard. When the steady density is lighter with increasing height, we show, with the help of a restricted condition imposed on steady density, that the steady-state is linearly globally stable and nonlinearly asymptotically stable in the sense of Hadamard.

* Corresponding author.
E-mail addresses: jiangfei0591@163.com (F. Jiang), jiang@iapcm.ac.cn (S. Jiang).

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1. Introduction

The motion of a three-dimensional (3D) nonhomogeneous incompressible viscous fluid in the presence of a uniform gravitational field in a bounded domain $\Omega \subset \mathbb{R}^3$ of C^2-class is governed by the following Navier–Stokes equations [21]:

$$\begin{cases} \rho_t + \mathbf{v} \cdot \nabla \rho = 0, \\ \rho \mathbf{v}_t + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mu \Delta \mathbf{v} - g \rho \mathbf{e}_3, \\ \operatorname{div} \mathbf{v} = 0, \end{cases} \tag{1.1}$$

where the unknowns $\rho := \rho(t, \mathbf{x})$, $\mathbf{v} := \mathbf{v}(t, \mathbf{x})$ and $p := p(t, \mathbf{x})$ denote the density, velocity and pressure of the fluid, respectively; $\mu > 0$ stands for the coefficient of shear viscosity, $g > 0$ for the gravitational constant, $\mathbf{e}_3 = (0, 0, 1)$ for the vertical unit vector, and $-g\mathbf{e}_3$ for the gravitational force. In the system (1.1) Eq. $(1.1)_1$ is the continuity equation, while $(1.1)_2$ describes the balance law of momentum.

The stability/instability of viscous incompressible flows governed by the Navier–Stokes equations is a classical subject with a very extensive literature over more than 100 years. In this paper we study the instability and stability of the following steady-state to the system (1.1):

$$\mathbf{v}(t, \mathbf{x}) \equiv \mathbf{0},$$
$$\nabla \bar{p} = -\bar{\rho} g \mathbf{e}_3 \quad \text{in } \Omega.$$

It is easy to show that the steady density $\bar{\rho}$ only depends on x_3, the third component of $\mathbf{x}$, provided $\bar{\rho} \in C^1(\Omega)$. Hence we denote $\bar{\rho}' := \partial_{x_3} \bar{\rho}$ in this paper for simplicity. Moreover, we can compute out the corresponding steady pressure $\bar{p}$ determined by $\bar{\rho}$. Now, denote the perturbation by

$$\varrho = \rho - \bar{\rho}, \qquad \mathbf{u} = \mathbf{v} - \mathbf{0}, \qquad q = p - \bar{p},$$

then, $(\varrho, \mathbf{u}, q)$ satisfies the perturbed equations:

$$\begin{cases} \varrho_t + \mathbf{u} \cdot \nabla (\varrho + \bar{\rho}) = 0, \\ (\varrho + \bar{\rho}) \mathbf{u}_t + (\varrho + \bar{\rho}) \mathbf{u} \cdot \nabla \mathbf{u} + \nabla q = \mu \Delta \mathbf{u} - g \varrho \mathbf{e}_3, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{1.2}$$

To complete the statement of the perturbed problem, we specify the initial and boundary conditions:

$$(\varrho, \mathbf{u})|_{t=0} = (\varrho_0, \mathbf{u}_0) \quad \text{in } \Omega \tag{1.3}$$

and

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0} \quad \text{for any } t > 0. \tag{1.4}$$

Moreover, the initial data should satisfy the compatibility condition $\operatorname{div} \mathbf{u}_0 = 0$.