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The Coolidge–Nagata conjecture, part I

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A R T I C L E   I N F O

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A B S T R A C T

Let $E \subseteq \mathbb{P}^2$ be a complex rational cuspidal curve contained in the projective plane and let $(X, D) \to (\mathbb{P}^2, E)$ be the minimal log resolution of singularities. Applying the log Minimal Model Program to $(X, \frac{1}{2}D)$ we prove that if E has more than two singular points or if D, which is a tree of rational curves, has more than six maximal twigs or if $\mathbb{P}^2 \setminus E$ is not of log general type then E is Cremona equivalent to a line, i.e. the Coolidge–Nagata conjecture for E holds. We show also that if E is not Cremona equivalent to a line then the morphism onto the minimal model contracts at most one irreducible curve not contained in D.

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1. Main results and strategy

Let $\bar{E} \subseteq \mathbb{P}^2$ be a complex planar rational curve which is *cuspidal*, i.e. which has only locally analytically irreducible (unibranched) singularities. Equivalently, it can be defined as an image of a singular embedding of a complex projective line into a complex projective plane, i.e. of a morphism $\mathbb{P}^1 \to \mathbb{P}^2$ which is 1–1 on closed points. We say that two planar curves are *Cremona equivalent* if one of them is a proper transform of the other under some Cremona transformation of $\mathbb{P}^2$. Not all rational curves on $\mathbb{P}^2$ are Cremona equivalent to a line (a general rational curve of degree at least six is not, see 2.6) and, clearly, the proper transform of $\bar{E}$ under a Cremona transformation does not have to be cuspidal. Therefore, the conjecture that nevertheless all cuspidal curves are Cremona equivalent to a line, which is known as the Coolidge–Nagata conjecture, comes as a surprise. It has been studied for a long time.[2] Let c be the number of cusps of $\bar{E}$ and let $(X, D) \to (\mathbb{P}^2, \bar{E})$ be the minimal log resolution of singularities. In [14] we proved that

$$c \leq 9 - 2p_2\big(\mathbb{P}^2, \bar{E}\big),$$

where $p_2(\mathbb{P}^2, \bar{E}) = h^0(2K_X + D)$. Let E be the proper transform of $\bar{E}$ on X. The Coolidge–Nagata conjecture for $\bar{E} \subseteq \mathbb{P}^2$ is known to be equivalent to the vanishing of $h^0(2K_X + E)$, so if it fails for $\bar{E}$ then we get a lower bound $p_2(\mathbb{P}^2, \bar{E}) \geq h^0(2K_X + E) \geq 1$. The higher lower bound on $p_2(\mathbb{P}^2, \bar{E})$ we can prove, the bigger is the restriction on c (in fact also on many other parameters describing the geometry of $\bar{E} \subseteq \mathbb{P}^2$), and hence the closer we are to proving the conjecture. Deepening the analysis of minimal models of $(X, \frac{1}{2}D)$ started in [14] (which is an analog of the 'theory of peeling' [8, §2.3] for half-integral divisors) we show here the following result.

Theorem 1.1. *Let $\bar{E} \subseteq \mathbb{P}^2$ be a complex rational cuspidal curve which is not Cremona equivalent to a line and let $(X, D) \to (\mathbb{P}^2, \bar{E})$ be the minimal log resolution of singularities. Then $p_2(\mathbb{P}^2, \bar{E}) \in \{3, 4\}$. Equivalently, $(K_X + D)^2 \in \{1, 2\}$.*

[2] Coolidge [2, Book IV, §II.2] and Nagata [11] studied planar rational curves and their behavior under the action of the Cremona group. The problem of determining which rational curves are Cremona equivalent to a line is known as the 'Coolidge–Nagata problem'.