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Reducts of the random partial order

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A R T I C L E I N F O

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A B S T R A C T

We determine, up to the equivalence of first-order interdefinability, all structures which are first-order definable in the random partial order. It turns out that there are precisely five such structures. We achieve this result by showing that there exist exactly five closed permutation groups which contain the automorphism group of the random partial order, and thus expose all symmetries of this structure.
Our result lines up with previous similar classifications for the random graph and the order of the rationals; it also provides further evidence for a conjecture due to Simon Thomas which states that the number of structures definable in a homogeneous structure in a finite relational language is, up to first-order interdefinability, always finite. In the proof we use the new technique of “canonical functions” originally invented in the context of theoretical computer science, which allows for a systematic Ramsey-theoretic analysis of functions acting on the random partial order. The technique identifies patterns in

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arbitrary functions on the random partial order, which makes them accessible to finite combinatorial arguments.

1. Reducts of homogeneous structures

The *random partial order* $\mathbb{P} := (P; \leq)$ is the unique countable partial order which is *universal* in the sense that it contains all countable partial orders as induced suborders and which is *homogeneous*, i.e., any isomorphism between two finite induced suborders of $\mathbb{P}$ extends to an automorphism of $\mathbb{P}$. Equivalently, $\mathbb{P}$ is the *Fraïssé limit* of the class of finite partial orders – confer the textbook [10].

As the "generic order" representing all countable partial orders, the random partial order is of both theoretical and practical interest. The latter becomes in particular evident with the recent applications of homogeneous structures in theoretical computer science; see for example [4,5,3,12]. It is therefore tempting to classify all structures which are first-order definable in $\mathbb{P}$, i.e., all relational structures on domain P all of whose relations can be defined from the relation $\leq$ by a first-order formula. Such structures have been called *reducts* of $\mathbb{P}$ in the literature [15,16,9]. It is the goal of the present paper to obtain such a classification *up to first-order interdefinability*, that is, we consider two reducts Γ, Γ' equivalent iff they are reducts of one another. We will show that up to this equivalence, there are precisely five reducts of $\mathbb{P}$.

Our result lines up with a number of previous classifications of reducts of similar generic structures up to first-order interdefinability; however, our proof uses for the first time the recent systematic method of *canonical functions*. The first non-trivial reduct classification was obtained by Cameron [8] for the order of the rationals, i.e., the Fraïssé limit of the class of finite linear orders; he showed that this order has five reducts up to first-order interdefinability. Thomas [15] proved that the random graph has five reducts up to first-order interdefinability as well, and later generalized this result by showing that for all $k \geq 2$, the random hypergraph with k-hyperedges has $2^k + 1$ reducts up to first-order interdefinability [16]. Junker and Ziegler [11] showed that the structure $(\mathbb{Q}; <, 0)$, i.e., the order of the rationals with an additional constant symbol, has 116 reducts up to interdefinability. Further examples include the random K_n-free graph for all $n \geq 3$ (2 reducts, see [15]), and the random tournament (5 reducts, see [1]). Obviously, the successful classifications have in common that the number of reducts is finite, and it is indeed an open conjecture of Thomas [15] that all homogeneous structures in a finite relational language have only finitely many reducts up to first-order interdefinability.

The mentioned classifications have all been obtained by means of the automorphism groups of the reducts, and we will proceed likewise in the present paper. It is clear that if Γ is a reduct of a structure Δ, then the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ is a permutation group containing $\mathrm{Aut}(\Delta)$, and also is a closed set with respect to the convergence topology on the space of all permutations on the domain of Δ. If Δ is