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Inflations of ideal triangulations

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ABSTRACT

Starting with an ideal triangulation of $\mathring{M}$, the interior of a compact 3-manifold M with boundary, no component of which is a 2-sphere, we provide a construction, called an inflation of the ideal triangulation, to obtain a strongly related triangulation of M itself. Besides a step-by-step algorithm for such a construction, we provide examples of an inflation of the two-tetrahedra ideal triangulation of the complement of the *figure-eight* knot in S^3, giving a minimal triangulation, having ten tetrahedra, of the *figure-eight* knot exterior. As another example, we provide an inflation of the one-tetrahedron Gieseking manifold giving a minimal triangulation, having seven tetrahedra, of a non-orientable compact 3-manifold with Klein bottle boundary. Several applications of inflations are discussed.

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1. Introduction

Triangulations play a central role in the study and understanding of 3-manifolds. They are used directly or indirectly for the major work on a census of 3-manifolds [3,12–14] and are fundamental to most of our advances on decision problems, algorithms, and issues of computational complexity. Triangulations naturally give rise to classes of surfaces called normal and almost normal surfaces, these surfaces in turn have been used in constructions of decompositions and recognition algorithms for 3-manifolds. A triangulation of a 3-manifold can be thought of as a combinatorial analog of a metric on the manifold and just as we try to deform metrics to gain geometric and topological information about a 3-manifold, we can similarly hope to gain geometric and topological information about a 3-manifold by deforming a given triangulation to a "good" triangulation of the 3-manifold. This work contributes to constructions that can be used to modify one triangulation to another that exhibits desirable properties.

It is well known that triangulations contain many normal surfaces that are not very interesting topologically but are artifacts of the triangulation; on the other hand, with certain modification, we can often arrive at a triangulation where there are useful connections between the geometry and topology of the manifold and the normal surfaces in the triangulation. In our work on 0-efficient triangulations [5], the aim was to control normal surfaces with positive Euler characteristic; this leads to a very nice algorithm for the connected sum decomposition of a 3-manifold [5] and triangulations that lend themselves nicely to the 3-sphere recognition algorithm [5,15,17]. For many algorithms and structure problems it is very desirable to control normal surfaces with zero Euler characteristic; our work on such triangulations includes the work presented here. One of its applications is control of normal annuli in 3-manifolds with boundary [7]. Angle structures in ideal triangulations give interesting examples of connections between normal surfaces and the geometry and topology of 3-manifolds. In fact the space of normal surfaces of an ideal triangulation forms a natural dual object of the space of angle structures [10,11].

In our study of positive Euler characteristic normal surfaces in a 3-manifold [5], we developed a technique for crushing a triangulation of a 3-manifold along a normal surface; in this work and subsequent work [7], we extend these techniques to manifolds with boundary, crushing a triangulation along the boundary (crushing the boundary to a point) and arriving at a related ideal triangulation of the interior of the 3-manifold. In our considerations of surface with zero Euler characteristic, we discovered an operation on ideal triangulations that is dual to the operation of crushing a triangulation of a 3-manifold with boundary along its boundary. We call this operation on an ideal triangulation an inflation of the ideal triangulation. Starting from an ideal triangulation of the interior of a compact 3-manifold with boundary, an inflation gives a strongly related triangulation of the compact 3-manifold itself, which, in turn, admits a crushing along its boundary returning to the original ideal triangulation.

In Section 3 of this paper, we review the construction of crushing a triangulation of a 3-manifold along a normal surface and apply these techniques to this work, which we