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A decomposition theorem for Herman maps

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ABSTRACT

In 1980s, Thurston established a topological characterization theorem for postcritically finite rational maps. In this paper, a decomposition theorem for a class of postcritically infinite branched covering termed Herman map is developed. It's shown that every Herman map can be decomposed along a stable multicurve into finitely many Siegel maps and Thurston maps, such that the combinations and rational realizations of these resulting maps essentially dominate the original one. This result is motivated by a non-expanding version of McMullen's problem, and Thurston's theory on characterization of rational maps. It enables us to prove a Thurston-type theorem for rational maps with Herman rings.

1. Introduction

In 1980s, Douady and Hubbard [10] revealed the complexity of the family of quadratic polynomials. Contemporaneously, Thurston's 3-dimensional insights revolutionized the theory of Kleinian group [38]. After then, Sullivan [35] discovered a dictionary between these two objects. Applying quasi-conformal method to rational maps [1], he translated the Ahlfors' finiteness theorem into a solution of a long-standing problem of wandering domains.

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Based on Sullivan's dictionary, McMullen asked a question: Is there a 3-dimensional geometric object naturally associated to a rational map? For example, it's known that Haken manifolds have a hierarchy, where they can be split up into 3-balls along incompressible surfaces. McMullen suggested to translate the Haken theory on cutting along general incompressible subsurfaces into a theory for rational maps with disconnected Julia sets. He posed the following problem [22, Problem 5.4]:

Problem 1.1 *(McMullen). Develop decomposition and combination theorems for expanding rational maps with disconnected Julia set.*

Here, the word *expanding* means hyperbolic. Recently, Guizhen Cui and Tan Lei [8] have developed the idea of *decomposition along a stable multicurve* and proven a characterization theorem for hyperbolic rational maps. Even if they didn't state their theorems in the decomposition and combination form, their powerful ideas and techniques give an answer to McMullen's problem in a sense.

In this article, motivated by this problem, we aim to develop a decomposition theorem for a kind of non-expanding rational maps: Rational maps with Herman rings, or more generally for *Herman maps*. Roughly speaking, a Herman map is a postcritically infinite branched covering with 'Herman rings' and postcritically finite outside the closure of all rotation domains. We will show that a Herman map can always be decomposed along a stable multicurve into two kinds of simpler maps — Siegel maps and Thurston maps — such that the combinations and rational realizations of these simpler maps essentially dominate the original one. Here, roughly, a Siegel map is a postcritically infinite branched covering with 'Siegel disks' and postcritically finite outside the closure of all these disks, a Thurston map is a postcritically finite branched covering. The precise formulation of the decomposition theorem requires a fair number of definitions and is put in the next section.

Another motivation of this work is Thurston's theory on characterization of rational maps. The theory deals with the following problem: Given a branched covering, when it is equivalent (in a proper sense) to a rational map? Thurston [39] gave a complete answer to this problem for postcritically finite cases in 1980s by showing that such a map either is equivalent to an essentially unique rational map or contains a Thurston obstruction. Here, an obstruction is a collection of Jordan curves such that a certain associated matrix has leading eigenvalue greater than 1. The detailed proof of Thurston's theorem is given by Douady and Hubbard [12] in 1993. The insights produce many new, sometimes unexpected applications in complex dynamics [3,4,6,8,13–15,19,26,30,29,28,31,34,36,37], etc. Since then, many people have tried to extend Thurston's theorem beyond postcritically finite rational maps. Recently, progress has made for several families of holomorphic maps. For example, Hubbard, Schleicher and Shishikura [19] extended Thurston's theorem to postsingularly finite exponential maps λe^z; Cui and Tan [8], Zhang and Jiang [44], independently, proved a Thurston-type theorem for hyperbolic rational