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On the stability of sets of even type ☆

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ABSTRACT

A stability theorem says that a nearly extremal object can be obtained from an extremal one by "small changes". In this paper, we prove a sharp stability theorem of sets of even type in $\mathrm{PG}(2,q)$, q even. As a consequence, we improve Blokhuis and Bruen's stability theorem on hyperovals and also on the minimum number of lines intersecting a point set of size at most $q+2\lfloor\sqrt{q}\rfloor-2$; furthermore we improve on the lower bound for untouchable sets.

1. Introduction

The main result of this paper is a stability theorem on sets of even type in $\mathrm{PG}(2,q)$, q even. A stability theorem says that when a structure is "close" to being extremal, then it can be obtained from an extremal one by changing it a little bit.

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A *set of even type* S is a point set intersecting each line in an even number of points. By counting the points of S on the lines through a point of S and on the lines through a point not in S, one can see immediately that q must be even. Hence from now on we will assume that $2|q$. Another motivation for studying sets of even type comes from coding theory. Such sets are codewords in the dual code of the code generated by the incidence matrix of $\mathrm{PG}(2,q)$, q even. The smallest sets of even type, corresponding to codewords of minimum weight, are the *hyperovals*; they have $q+2$ points. Somewhat larger sets of even type were constructed by Korchmáros and Mazzocca [18]. A $(q+t,t)$-arc of type $(0,2,t)$ in $\mathrm{PG}(2,q)$ is a set S of $q+t$ points such that every line meets S in either $0,2$ or t points. It is known, see [18], that t has to be a divisor of q. Korchmáros and Mazzocca also conjecture that whenever 4 divides t and t divides q, then there exists a $(q+t,t)$-arc of type $(0,2,t)$. Several constructions can be found in [18] and in Gács, Weiner [14], where it is also proved that the t-secants of a $(q+t,t)$-arc of type $(0,2,t)$ are concurrent. Among the sporadic examples we mention the $(36,4)$-arcs of type $(0,2,4)$ in $\mathrm{PG}(2,32)$ found by Key, McDonough, and Mavron [17]. More examples are given by Limbupasiriporn [20,21]. Recently, a new infinite class of $(q+t,t)$-arcs of type $(0,2,t)$ was constructed by Vandendriessche for $t=q/4$, see [23]. Planar sets of even type also appear in classifying small weight codewords of the dual code generated by characteristic vectors of hyperplanes of $\mathrm{PG}(n,q)$, see De Boeck [10].

A set of *almost even type* is a point set having only few odd-secants. If we delete a point from a set of even type S then the new point set will have $q+1$ odd-secants, and of course this will also be the case when we add a point to S. If ε points are modified then at least $\varepsilon(q+1-(\varepsilon-1))$ and at most $\varepsilon(q+1)$ odd-secants are obtained. A further motivation to study sets of almost even type comes from small Kakeya-sets in $\mathrm{AG}(2,q)$, q even. If the Kakeya-set is small, then the lines of the Kakeya-set cover almost all points twice, hence in the dual plane they give sets of almost even type. For more details, see [3] and the recent paper [7].

The main result of this paper is the following stability theorem, which will be proved in Section 3.

Theorem 1.1. *Assume that the point set* $\mathcal{M}$ *in* $\mathrm{PG}(2,q)$*,* $16<q$ *even, has* δ *odd-secants, where* $\delta<(\lfloor\sqrt{q}\rfloor+1)(q+1-\lfloor\sqrt{q}\rfloor)$*. Then there exists a unique set* $\mathcal{M}'$ *of even type, such that* $|(\mathcal{M}\cup\mathcal{M}')\setminus(\mathcal{M}\cap\mathcal{M}')|=\lceil\frac{\delta}{q+1}\rceil$*.*

Let us also interpret this result in terms of codes. A set of points corresponds to a set of lines in the dual plane. Odd-secants correspond to points that are contained in an odd number of lines. Considering the sum of the characteristic vectors of the original lines these are just codewords in the p-ary linear code $(C_1(2,q))$ generated by lines of $\mathrm{PG}(2,q)$, $q=p^h$. If the number of odd-secants is δ in the original setting, then this codeword has weight (the number of non-zero coordinates) δ. Hence the above result has the following corollary.