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Quantum symmetric Kac–Moody pairs

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ABSTRACT

The present paper develops a general theory of quantum group analogs of symmetric pairs for involutive automorphism of the second kind of symmetrizable Kac–Moody algebras. The resulting quantum symmetric pairs are right coideal subalgebras of quantized enveloping algebras. They give rise to triangular decompositions, including a quantum analog of the Iwasawa decomposition, and they can be written explicitly in terms of generators and relations. Moreover, their centers and their specializations are determined. The constructions follow G. Letzter's theory of quantum symmetric pairs for semisimple Lie algebras. The main additional ingredient is the classification of involutive automorphisms of the second kind of symmetrizable Kac–Moody algebras due to Kac and Wang. The resulting theory comprises various classes of examples which have previously appeared in the literature, such as q-Onsager algebras and the twisted q-Yangians introduced by Molev, Ragoucy, and Sorba.

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1. Introduction

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra defined over an algebraically closed field $\mathbb{K}$ of characteristic 0. The quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$, discovered by Drinfeld [15] and Jimbo [25] nearly thirty years ago, is an integral part of representation theory with many deep applications. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive Lie algebra automorphism and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into the $(+1)$- and the (-1)-eigenspace of θ. The present paper is concerned with the construction and the structure theory of quantum group analogs of $U(\mathfrak{k})$ as right coideal subalgebras $B = B(\theta)$ of $U_q(\mathfrak{g})$. We call such analogs quantum symmetric pair coideal subalgebras and refer to $(U_q(\mathfrak{g}), B)$ as a quantum symmetric pair.

If $\mathfrak{g}$ is finite dimensional then there exist two approaches to the construction of quantum symmetric pairs. In the early nineties, Noumi, Sugitani, and Dijkhuizen constructed quantum group analogs of $U(\mathfrak{k})$ as coideal subalgebras of $U_q(\mathfrak{g})$ for all $\mathfrak{g}$ of classical type [47–49]. Their approach is based on explicit solutions of the reflection equation [10,35] and hence follows in spirit the methods developed by the (then) Leningrad school of mathematical physics [17]. Independently, G. Letzter developed a comprehensive theory of quantum symmetric pairs for all semisimple symmetric Lie algebras [36–42]. Her theory is based on the Drinfeld–Jimbo presentation of quantized enveloping algebras. It is well understood that the two approaches to quantum symmetric pairs provide essentially the same coideal subalgebras of $U_q(\mathfrak{g})$, see [37, Section 6].

Over the last decade, numerous examples of quantum symmetric pairs for infinite dimensional symmetrizable Kac–Moody algebras have appeared in the literature. Here we group these examples in three classes.

(1) *q*-Onsager algebras: The q-Onsager algebra is a quantum symmetric pair coideal subalgebra for the Chevalley involution of the affine Lie algebra $\widehat{\mathfrak{sl}}_2(\mathbb{K})$. It derives its name from the fact that the Lie subalgebra of $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ fixed under the Chevalley involution appeared in Onsager's investigation of the planar Ising model [51], see also [58, Remark 9.1] for historical comments. The q-Onsager algebra appeared, as an algebra, in