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Limit distributions of random matrices

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ABSTRACT

We study limit distributions of independent random matrices as well as limit joint distributions of their blocks under normalized partial traces composed with classical expectation. In particular, we are concerned with the ensemble of symmetric blocks of independent Hermitian random matrices which are asymptotically free, asymptotically free from diagonal deterministic matrices, and whose norms are uniformly bounded almost surely. This class contains symmetric blocks of unitarily invariant Hermitian random matrices whose asymptotic distributions are compactly supported probability measures on the real line. Our approach is based on the concept of matricial freeness which is a generalization of freeness in free probability. We show that the associated matricially free Gaussian operators provide a unified framework for studying the limit distributions of sums and products of independent rectangular random matrices, including non-Hermitian Gaussian matrices and matrices of Wishart type.

1. Introduction and main results

One of the most important features of free probability is its close relation to random matrices. It has been shown by Voiculescu [33] that Hermitian random matrices with

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independent Gaussian entries are asymptotically free. This result has been generalized by Dykema to non-Gaussian random matrices [12] and has been widely used by many authors in their studies of asymptotic distributions of random matrices. It shows that there is a concept of noncommutative independence, called *freeness*, which is fundamental to the study of large random matrices and puts the classical result of Wigner [36] on the semicircle law as the limit distribution of certain symmetric random matrices in an entirely new perspective.

In particular, if we are given an ensemble of independent Hermitian $n \times n$ random matrices

$$\{Y(u,n) : u \in \mathcal{U}\}$$

whose entries are suitably normalized and independent complex Gaussian random variables for each natural n, then

$$\lim_{n\to\infty} \tau(n)\big(Y(u_1,n)\ldots Y(u_m,n)\big) = \Phi\big(\omega(u_1)\ldots\omega(u_m)\big),$$

for any $u_1,\ldots,u_m \in \mathcal{U}$, where $\{\omega(u) : u \in \mathcal{U}\}$ is a semicircular family of *free Gaussian operators* living in the free Fock space with the vacuum state Φ and $\tau(n)$ is the normalized trace composed with classical expectation called the *trace* in the sequel. This realization of the limit distribution gives a fundamental relation between random matrices and operator algebras.

The basic original random matrix model studied by Voiculescu corresponds to independent complex Gaussian variables, where the entries $Y_{i,j}(u,n)$ of each matrix $Y(u,n)$ satisfy the Hermiticity condition, have mean zero, and the variances of real-valued diagonal Gaussian variables $Y_{j,j}(u,n)$ are equal to $1/n$, whereas those of the real and imaginary parts of the off-diagonal (complex-valued) Gaussian variables $Y_{i,j}(u,n)$ are equal to $1/2n$. If we relax the assumption on equal variances, the scalar-valued free probability is no longer sufficient to describe the asymptotics of Gaussian random matrices. One approach is to use the operator-valued free probability, as in the work of Shlyakhtenko [32], who studied the asymptotics of Gaussian random band matrices and proved that they are asymptotically free with amalgamation over some commutative algebra. This approach was further developed by Benaych-Georges [6], who treated the case when one of the block asymptotic dimensions vanishes.

Our approach is based on the decomposition of independent Hermitian Gaussian random matrices $Y(u,n)$ with block-identical variances of $|Y_{i,j}(u,n)|$ into symmetric blocks $T_{p,q}(u,n)$, where $u \in \mathcal{U}$ and $n \in \mathbb{N}$, namely

$$Y(u,n) = \sum_{1\le p\le q\le r} T_{p,q}(u,n),$$

where symmetric blocks can be written in the form