Advances in Mathematics 263 (2014) 412–467

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Stability of hyperbolic manifolds with cusps under Ricci flow

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A R T I C L E   I N F O

Article history:
Received 7 June 2013
Accepted 6 July 2014
Available online 19 July 2014
Communicated by Gang Tian

Keywords:
Ricci flow
Stability
Hyperbolic manifold
Hyperbolic cusp
Cusp deformation

A B S T R A C T

We show that every finite volume hyperbolic manifold of dimension greater than or equal to 3 is stable under rescaled Ricci flow, i.e. that every small perturbation of the hyperbolic metric flows back to the hyperbolic metric again. Note that we do not need to make any decay assumptions on this perturbation.

It will turn out that the main difficulty in the proof comes from a weak stability of the cusps which has to do with infinitesimal cusp deformations. We will overcome this weak stability by using a new analytical method developed by Koch and Lamm.

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http://dx.doi.org/10.1016/j.aim.2014.07.003
0001-8708/

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1. Introduction and statement of the result

In this paper, we will prove the following theorem

Theorem 1.1. *For any complete hyperbolic manifold $(M^n, \bar{g})$ of finite volume and dimension $n \geq 3$, there is an $\varepsilon > 0$ such that the following holds:*

If g_0 is another smooth metric on M with

$$(1 - \varepsilon)\bar{g} \leq g_0 \leq (1 + \varepsilon)\bar{g},$$

then there is a solution $(g_t)_{t\in[0,\infty)}$ to the rescaled Ricci flow equation

$$\dot{g}_t = -2\mathrm{Ric}_{g_t} - 2(n-1)g_t$$

starting from g_0 which exists for all time and as $t \to \infty$ we have convergence $g_t \to \bar{g}$ in the pointed smooth Cheeger–Gromov sense, i.e. there is a family of diffeomorphisms Ψ_t of M such that $\Psi_t^* g_t \to \bar{g}$ in the smooth sense on every compact subset of M.

Moreover, ε can be chosen so that it only depends on an upper volume bound on M for $n \geq 4$ resp. an upper diameter bound on the compact part M_{cpt} of M for $n = 3$ (see Section 2.1 for more details).

Observe, that the theorem is already known in the *compact* case (see e.g. [26]). The *finite volume* case is more general than the compact case since it allows the manifold to have cusps, and hence to be noncompact (for a geometric description of these manifolds see Section 2.1). A similar stability result also holds in dimension 2. However, one has to take into account a finite dimensional deformation space of the hyperbolic structure