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The fifth coefficient for bounded univalent functions with real coefficients

Eugene Rodemich

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ABSTRACT

All stationary values of the fifth coefficient of bounded univalent functions with real coefficients are found, including maximum and minimum values.

1. Introduction

The class of functions considered here consists of the analytic functions

$$f(z) = \sum_{n=1}^{\infty} c_n z^n, \tag{1}$$

univalent in $\{|z| < 1\}$, with real coefficients and

$$\left|f(z)\right| < 1, \qquad f(0) = 0, \qquad f'(0) = c_1 > 0. \tag{2}$$

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In this paper we find the functions for which the fifth coefficient c_5 has a local extremum under variations of the function which preserve the value of c_1. The results include the maximum and minimum values of c_5 for any fixed c_1. The coefficient c_5 was treated rather than some other c_n because dealing with later coefficients would involve hyperelliptic integrals, needing numerical integration.

2. The differential equation

The functions sought are solutions of a differential equation which can be obtained by using an appropriate boundary variation of the image region. Schiffer [22,23] used the boundary variation

$$\delta w = \frac{\epsilon w}{w - w_0}, \tag{3}$$

where w_0 is a point not on the boundary and ϵ is a small complex constant. This variation leads to functions near the original in the family of univalent functions in the unit circle with $f(0) = 0$. Other writers [3,5–21,24–32] have adapted this variation to various families of univalent functions.

To preserve the conditions that $|f(z)| < 1$ and the coefficients are real, the variation (3) can be modified by including additional terms. We take

$$\delta w = \frac{\epsilon w}{w - w_0} + \frac{\bar{\epsilon} w}{w - \overline{w_0}} - \frac{\bar{\epsilon} w^2}{1 - \overline{w_0} w} - \frac{\epsilon w^2}{1 - w_0 w}. \tag{4}$$

The second term preserves real coefficients. The last two terms make $\delta w/w$ pure imaginary if $|w| = 1$. Applying this variation, the methods of [22] lead to the following differential equation for a function for which c_5 is extremal:

$$S(f) \frac{z^2 f'^2}{f^2} = Q(z), \tag{5}$$

where

$$\begin{aligned} S(f) = {} & \frac{c_1^5}{f^4} + \frac{4c_1^3 c_2}{f^3} + \frac{3(c_1^2 c_3 + c_1 c_2^2)}{f^2} + \frac{2(c_1 c_4 + c_2 c_3)}{f} + c_5 - \lambda c_1 \\ & + 2(c_1 c_4 + c_2 c_3) f + 3\left(c_1^2 c_3 + c_1 c_2^2\right) f^2 + 4c_1^3 c_2 f^3 + c_1^5 f^4, \end{aligned} \tag{6}$$

$$Q(z) = \frac{c_1}{z^4} + \frac{2c_2}{z^3} + \frac{3c_3}{z^2} + \frac{4c_4}{z} + 5c_5 - \lambda c_1 + 4c_4 z + 3c_3 z^2 + 2c_2 z^3 + c_1 z^4. \tag{7}$$

Here λ is a Lagrange multiplier for the side condition that c_1 is fixed.

The extremal functions map the unit disc onto the unit disc with slits. The function $Q(z)$ is real on the unit circle, non-negative for a local maximum of c_5, non-positive for a local minimum. Each endpoint of the slit system corresponds to a double zero of $Q(z)$