

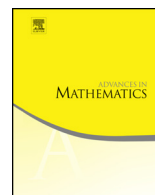


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# The fifth coefficient for bounded univalent functions with real coefficients

Eugene Rodemich

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## ABSTRACT

All stationary values of the fifth coefficient of bounded univalent functions with real coefficients are found, including maximum and minimum values.

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## 1. Introduction

The class of functions considered here consists of the analytic functions

$$f(z) = \sum_{n=1}^{\infty} c_n z^n, \quad (1)$$

univalent in  $\{|z| < 1\}$ , with real coefficients and

$$|f(z)| < 1, \quad f(0) = 0, \quad f'(0) = c_1 > 0. \quad (2)$$

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In this paper we find the functions for which the fifth coefficient  $c_5$  has a local extremum under variations of the function which preserve the value of  $c_1$ . The results include the maximum and minimum values of  $c_5$  for any fixed  $c_1$ . The coefficient  $c_5$  was treated rather than some other  $c_n$  because dealing with later coefficients would involve hyperelliptic integrals, needing numerical integration.

**2. The differential equation**

The functions sought are solutions of a differential equation which can be obtained by using an appropriate boundary variation of the image region. Schiffer [22,23] used the boundary variation

$$\delta w = \frac{\epsilon w}{w - w_0}, \tag{3}$$

where  $w_0$  is a point not on the boundary and  $\epsilon$  is a small complex constant. This variation leads to functions near the original in the family of univalent functions in the unit circle with  $f(0) = 0$ . Other writers [3,5–21,24–32] have adapted this variation to various families of univalent functions.

To preserve the conditions that  $|f(z)| < 1$  and the coefficients are real, the variation (3) can be modified by including additional terms. We take

$$\delta w = \frac{\epsilon w}{w - w_0} + \frac{\bar{\epsilon} w}{w - \bar{w}_0} - \frac{\bar{\epsilon} w^2}{1 - \bar{w}_0 w} - \frac{\epsilon w^2}{1 - w_0 w}. \tag{4}$$

The second term preserves real coefficients. The last two terms make  $\delta w/w$  pure imaginary if  $|w| = 1$ . Applying this variation, the methods of [22] lead to the following differential equation for a function for which  $c_5$  is extremal:

$$S(f) \frac{z^2 f'^2}{f^2} = Q(z), \tag{5}$$

where

$$S(f) = \frac{c_1^5}{f^4} + \frac{4c_1^3 c_2}{f^3} + \frac{3(c_1^2 c_3 + c_1 c_2^2)}{f^2} + \frac{2(c_1 c_4 + c_2 c_3)}{f} + c_5 - \lambda c_1 + 2(c_1 c_4 + c_2 c_3) f + 3(c_1^2 c_3 + c_1 c_2^2) f^2 + 4c_1^3 c_2 f^3 + c_1^5 f^4, \tag{6}$$

$$Q(z) = \frac{c_1}{z^4} + \frac{2c_2}{z^3} + \frac{3c_3}{z^2} + \frac{4c_4}{z} + 5c_5 - \lambda c_1 + 4c_4 z + 3c_3 z^2 + 2c_2 z^3 + c_1 z^4. \tag{7}$$

Here  $\lambda$  is a Lagrange multiplier for the side condition that  $c_1$  is fixed.

The extremal functions map the unit disc onto the unit disc with slits. The function  $Q(z)$  is real on the unit circle, non-negative for a local maximum of  $c_5$ , non-positive for a local minimum. Each endpoint of the slit system corresponds to a double zero of  $Q(z)$

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