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The universal enveloping algebra of the Witt algebra is not noetherian

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A B S T R A C T

This work is prompted by the long standing question of whether it is possible for the universal enveloping algebra of an infinite dimensional Lie algebra to be noetherian. To address this problem, we answer a 23-year-old question of Carolyn Dean and Lance Small; namely, we prove that the universal enveloping algebra of the Witt (or centerless Virasoro) algebra is not noetherian. To show this, we prove our main result: the universal enveloping algebra of the positive part of the Witt algebra is not noetherian. We employ algebro-geometric techniques from the first author's classification of (noncommutative) birationally commutative projective surfaces.

As a consequence of our main result, we also show that the enveloping algebras of many other (infinite dimensional) Lie algebras are not noetherian. These Lie algebras include the Virasoro algebra and all infinite dimensional $\mathbb{Z}$-graded simple Lie algebras of polynomial growth.

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0. Introduction

To begin, we take $\Bbbk$ to be a field of characteristic 0 and we let an unadorned $\otimes$ mean $\otimes_{\Bbbk}$. (If X is a scheme, and $\mathcal{A}$, $\mathcal{B}$ are quasicoherent sheaves on X, we write $\mathcal{A} \otimes_X \mathcal{B}$ rather than $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}$.) We are motivated by the well-known question of whether it is possible for the universal enveloping algebra of an infinite dimensional Lie algebra to be noetherian (cf. [7, p. xix]). It is generally thought that the answer to this question should be "no," and we state this as a conjecture:

Conjecture 0.1. *A Lie algebra L is finite dimensional if and only if the universal enveloping algebra $U(L)$ is noetherian.*

One direction holds easily. Namely, the universal enveloping algebra of a finite dimensional Lie algebra is noetherian; see [13, Corollary 1.7.4]. To address the converse, many have considered the universal enveloping algebra of the infinite dimensional Lie algebra W below.

Definition 0.2. $[W, U(W)]$ The *Witt (or centerless Virasoro) algebra* W is defined to be the Lie algebra W with basis $\{e_n\}_{n\in\mathbb{Z}}$ and Lie bracket $[e_n, e_m] = (m-n)e_{n+m}$. We take $U(W)$ to be the universal enveloping algebra of W, which is $\mathbb{Z}$-graded with $\deg(e_n) = n$.

Note that W is realized as the Lie algebra of derivations of $\Bbbk[x, x^{-1}]$, where $e_n = x^{n+1}\frac{d}{dx}$. If $\Bbbk = \mathbb{C}$, then W is also the complexification of the Lie algebra of polynomial vector fields on the circle. Here, $e_n = -i\exp(in\theta)\frac{d}{d\theta}$, where θ is the angular parameter.

It is well known that $U(W)$ is a domain, has infinite global dimension, and has subexponential growth [3, Section 3]. On the other hand, regarding Conjecture 0.1, we have:

Question 0.3. (C. Dean and L. Small, 1990.) Is $U(W)$ noetherian?

We consider the following subalgebra of $U(W)$, which aids in answering Question 0.3 above.

Definition 0.4. $[W_+, U(W_+)]$ The *positive (part of the) Witt algebra* is defined to be the Lie subalgebra W_+ of W generated by $\{e_n\}_{n\geq 1}$. The universal enveloping algebra $U(W_+)$ is then the following subalgebra of $U(W)$:

$$U(W_+) = \frac{\Bbbk\langle e_n \mid n \geq 1\rangle}{([e_n, e_m] = (m-n)e_{n+m})},$$

which is $\mathbb{N}$-graded with $\deg e_n = n$.

It is a general fact that if L' is a Lie subalgebra of L and $U(L)$ is noetherian, then $U(L')$ is noetherian; see Lemma 1.7. Because of this, many have asked whether $U(W_+)$ is noetherian. We show that it is not, thus answering Question 0.3 as follows.