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Graded quiver varieties, quantum cluster algebras and dual canonical basis

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ABSTRACT

Inspired by a previous work of Nakajima, we consider perverse sheaves over acyclic graded quiver varieties and study the Fourier–Sato–Deligne transform from a representation theoretic point of view. We obtain deformed monoidal categorifications of acyclic quantum cluster algebras with specific coefficients. In particular, the (quantum) positivity conjecture is verified whenever there is an acyclic seed in the (quantum) cluster algebra.

In the second part of the paper, we introduce new quantizations and show that all quantum cluster monomials in our setting belong to the dual canonical basis of the corresponding quantum unipotent subgroup. This result generalizes previous work by Lampe and by Hernandez–Leclerc from the Kronecker and Dynkin quiver case to the acyclic case.

The Fourier transform part of this paper provides crucial input for the second author's paper where he constructs bases of acyclic quantum cluster algebras with arbitrary coefficients and quantization.

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1. Introduction

1.1. Motivation

Cluster algebras were invented by Fomin and Zelevinsky in [8]. They are algebras generated by certain combinatorially defined generators (the cluster variables). The quantum deformations were defined in [3]. Fomin and Zelevinsky stated their original motivation as follows:

This structure should serve as an algebraic framework for the study of dual canonical bases in these coordinate rings and their q-deformations. In particular, we conjecture that all monomials in the variables of any given cluster (the cluster monomials) belong to this dual canonical basis.

However, despite the many successful applications of (quantum) cluster algebras to other areas (*cf.* the introductory survey by Bernhard Keller [18]), the link between (quantum) cluster monomials and the dual canonical basis of quantum groups remains largely open. Partial results are due to [21,22,14] for quivers of finite and affine type.

Also, the following conjecture has attracted a lot of interest since the invention of cluster algebras.

Conjecture 1.1.1 *(Positivity conjecture). With respect to the cluster variables in any given seed, each cluster variable expands into a Laurent polynomial with non-negative integer coefficients.*

This conjecture has been proved for cluster algebras arising from surfaces by Gregg Musiker, Ralf Schiffler, and Lauren Williams [26], for cluster algebras containing a bipartite seed by Nakajima [30], and the quantized version for quantum cluster algebras with respect to an acyclic initial seed by [34]. Recently, Efimov obtained further partial results on this conjecture for quantum cluster algebras containing an acyclic seed using mixed Hodge modules, *cf.* [7]. After this article was posted on arxiv, Kyungyong Lee and Ralf Schiffler informed the authors about a combinatorial