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On the Lie algebroid of a derived self-intersection

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A B S T R A C T

Let $i : X \hookrightarrow Y$ be a closed embedding of smooth algebraic varieties. Denote by N the normal bundle of X in Y. The present paper contains two constructions of certain Lie structure on the shifted normal bundle $N[-1]$ encoding the information of the formal neighborhood of X in Y. We also present a few applications of these Lie theoretic constructions in understanding the algebraic geometry of embeddings.

Contents

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1. Introduction

The aim of this paper is to study the derived self-intersection of a closed X into Y, when X and Y are smooth algebraic varieties. The word "derived" appears here because self-intersections are badly-behaved (e.g. they are not transverse and their actual dimension does not coincide with the expected one). If X and Y were to be differentiable manifolds we could consider an appropriate small perturbation X_ϵ of X and take $X \cap X_\epsilon$, but this approach has several drawbacks:

- it does not preserve the set-theoretical intersection;
- it does lead to some category theoretic problems (no functorial choice for X_ϵ);
- we can't do this with algebraic varieties.

It has been known for a long time that a possible replacement for actual geometric perturbation is homological perturbation. More precisely, if $i : X \hookrightarrow Y$ is a closed embedding then we will consider a resolution $\mathcal{R}$ of $\mathcal{O}_X$ by a differential graded (dg) $i^{-1}\mathcal{O}_Y$-algebra, and equip X with the sheaf of dg-rings $\mathcal{O}_X \otimes^{\mathbb{L}}_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X := \mathcal{R} \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X$. The pair $(X, \mathcal{O}_X \otimes^{\mathbb{L}}_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X)$ is a dg-scheme that we will denote $X \times^h_Y X$ and call the derived self-intersection of X into Y (or more generally the self homotopy fiber product of X over Y for a general morphism $X \to Y$).

Observe that different choices of resolutions give rise to weakly equivalent results, and that there exist functorial choices for such resolutions.

In the rest of the introduction we allow ourselves to deal informally with schemes as if they would be topological spaces, where resolutions have to be thought as fibrant replacements.[2]

1.1. The diagonal embedding (after M. Kapranov)

Let us consider the example of a diagonal embedding $\Delta : X \hookrightarrow X \times X$. One way to compute the homotopy fiber product is to factor $\Delta : X \to X \times X$ into an acyclic cofibration followed by a fibration: $X \overset{\sim}{\rightarrowtail} \tilde{X} \twoheadrightarrow X \times X$. This is achieved by taking $\tilde{X} := \{\gamma : [0,1] \to X \times X \,|\, \gamma(0) \in \Delta(X)\}$ and $\gamma \mapsto \gamma(1)$ as fibration. Therefore the derived self-intersection of the diagonal is

$$\tilde{X} \times_{X \times X} X = \{\gamma : [0,1] \to X \times X \,|\, \gamma(0), \gamma(1) \in \Delta(X)\},$$

which happens to be weakly equivalent to the loop space of X (*via* the map sending a path γ in $X \times X$ with both ends in the diagonal to the loop $\tilde{\gamma}$ in X defined by $\tilde{\gamma}(t) = \begin{cases} \pi_1(\gamma(2t)) \text{ if } 0 \le t \le 1/2 \\ \pi_2(\gamma(2-2t)) \text{ if } 1/2 \le t \le 1 \end{cases}$).

[2] This can be made precise using model categories and the advanced technology of homotopical and derived algebraic geometry, after Toën–Vezzosi [18,19] and Lurie [14]. Despite the fact that this is certainly the appropriate framework to work with, we will nevertheless stay within the realm of dg-schemes (after [6]), which are sufficient for our purposes.