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Projective planes over 2-dimensional quadratic algebras

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ABSTRACT

The split version of the Freudenthal–Tits magic square stems from Lie theory and constructs a Lie algebra starting from two split composition algebras [5,20,21]. The geometries appearing in the second row are Severi varieties [24]. We provide an easy uniform axiomatization of these geometries and related ones, over an arbitrary field. In particular we investigate the entry $A_2 \times A_2$ in the magic square, characterizing Hermitian Veronese varieties, Segre varieties and embeddings of Hjelmslev planes of level 2 over the dual numbers. In fact this amounts to a common characterization of "projective planes over 2-dimensional quadratic algebras", in cases of the split and non-split Galois extensions, the inseparable extensions of degree 2 in characteristic 2 and the dual numbers.

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1. Introduction, notation and main result

1.1. Mazzocca–Melone axioms and $\mathcal{C}$-Veronesean sets

In this paper we present a far-going generalization of the Mazzocca–Melone approach to quadric Veronese varieties. In this introduction, we describe the formal situation, and mention some history. The compelling motivation for our approach (why exactly generalizing in the way we do) is explained in the final section of the paper in order not to interfere with its mathematical flow. The reader might want to read Section 6 first. It puts our result in the broader perspective of the Freudenthal–Tits Magic Square [5, 20,21], certain quadratic alternative algebras, and representations of a class of spherical buildings (containing those having exceptional type E_6) in projective space. Let us just mention here that our intention is not just "generalizing", but rather a new geometric approach to the aforementioned magic square.

The Mazzocca–Melone axioms for Veronese varieties have proved to be of fundamental importance for the theory of Veronese varieties. Mazzocca and Melone [10] were the first ones to prove such a characterization (for quadric Veronese varieties of finite projective planes), and the same axioms, with only some minor changes depending on the context, were used by others to characterize finite quadric Veronese varieties of projective spaces [17], quadric Veronese varieties in general [13], finite Hermitian Veronese varieties [18], and Hermitian Veronese varieties in general [12]. In this paper, we introduce a further minor change in these axioms to include more varieties over arbitrary fields.

Let us start by briefly recalling the "classical" Mazzocca–Melone axioms, in their simplest form, namely, for the quadric Veronesean variety X of a projective plane $\mathbb{P}^2(\mathbb{K})$ in the 5-dimensional projective space $\mathbb{P}^5(\mathbb{K})$ (see [13]), given by the image of the Veronese map $(x, y, z) \mapsto (x^2, y^2, z^2, yz, zx, xy)$. One hypothesizes a family $\mathit{\Pi}$ of planes of $\mathbb{P}^5(\mathbb{K})$ each member of which intersects X in a conic (or, more generally, an *oval*, see below). The set X corresponds to the point set of $\mathbb{P}^2(\mathbb{K})$, whereas the conics correspond to the lines of $\mathbb{P}^2(\mathbb{K})$. The pair $(X, \mathit{\Pi})$ satisfies three properties. The first is that every pair of points in X is contained in a unique member of $\mathit{\Pi}$; the second is that every pair of planes in $\mathit{\Pi}$ intersects inside X; the third is that the tangent lines at any point $x \in X$ to the conics obtained from $\mathit{\Pi}$ and going through x are contained in a plane only depending on x. The axioms for a Hermitian Veronesean are the same, except that $\mathit{\Pi}$ is replaced by a family of 3-spaces each member of which intersects the point set X in a quadric of Witt index 1. One possible way to go on would be to replace 3-spaces with n-spaces (see [8] for such an approach). Another generalization is to consider other classes of quadrics in 3-space. That is exactly what we will do. The achievement is then that the corresponding varieties precisely correspond in a uniform and explicit way to projective planes over 2-dimensional unital algebras.

So our first aim is to write down a satisfactory new version of the classical Mazzocca–Melone axioms that capture exactly the varieties we are aiming at. Two features have to be taken into account. The first one is that the quadrics no longer have Witt index 1;