Advances in Mathematics 262 (2014) 1002–1034

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Unitary equivalence of automorphisms of separable C*-algebras ☆

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ARTICLE INFO

Article history:
Received 26 April 2013
Accepted 28 May 2014
Available online 24 June 2014
Communicated by Dan Voiculescu

MSC:
03E15
46L40
46L57

Keywords:
C*-algebras
Automorphisms
Borel complexity
Descriptive set theory

ABSTRACT

We prove that the automorphisms of any separable C*-algebra that does not have continuous trace are not classifiable by countable structures up to unitary equivalence. This implies a dichotomy for the Borel complexity of the relation of unitary equivalence of automorphisms of a separable unital C*-algebra: Such relation is either smooth or not even classifiable by countable structures.

☆ The research was supported by the York University Elia Scholars Program and by the Oberwolfach Leibniz Graduate Student Programme.

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http://dx.doi.org/10.1016/j.aim.2014.05.022
0001-8708/

1. Introduction

If A is a separable C*-algebra, the group $\mathrm{Aut}(A)$ of automorphisms of A is a Polish group with respect to the topology of pointwise norm convergence. An automorphism of A is called (multiplier) *inner* if it is induced by the action by conjugation of a unitary element of the multiplier algebra $M(A)$ of A. Inner automorphisms form a Borel normal subgroup $\mathrm{Inn}(A)$ of the group of automorphisms of A. The relation of *unitary equivalence* of automorphisms of A is the coset equivalence relation on $\mathrm{Aut}(A)$ determined by $\mathrm{Inn}(A)$. (The reader can find more background on C*-algebras in Section 2.) The main result presented here asserts that if A does not have continuous trace, then it is not possible to effectively classify the automorphisms of A up to unitary equivalence using countable structures as invariants; in particular this rules out classification by K-theoretic invariants. (The K-theoretic invariants of C*-algebras were shown to be computable by a Borel function in [12, Theorem 3.3]. Even though [12, Theorem 3.3] does not consider the K-theory of $*$-homomorphisms, it is not difficult to verify that the proof can be adapted to show that the computation of K-theory of $*$-homomorphisms is given by a *Borel functor*. The main ingredient of the proof is the fact that one can enumerate in a Borel fashion dense sequences of projections and of unitary elements of the algebra and of all its amplifications [13, Lemma 3.13].) In the course of the proof of the main result we will show that the existence of an outer derivation on a C*-algebra A is equivalent to a seemingly stronger statement, that we will refer to as Property AEP (see Definition 4.4), implying in particular the existence of an outer derivable automorphism of A.

The notion of effective classification can be made precise by means of Borel reductions in the framework of descriptive set theory (the monographs [21] and [15] are standard references for this subject). If E and E' are equivalence relations on standard Borel spaces X and X', respectively, then a Borel reduction from E to F is a Borel function $f : X \to X'$ such that for every $x, y \in X$, xEy if and only if $f(x)E'f(y)$. The Borel function f witnesses an *effective classification* of the objects of X up to E, with E'-equivalence classes of objects of X' as invariants. This framework captures the vast majority of concrete classification results in mathematics. (In [13] and [12] the computation of most of the invariants in the theory of C*-algebras is shown to be Borel.)

If E and F are, as before, equivalence relations on standard Borel spaces, then E is *Borel reducible* to F if there is a Borel reduction from E to F. This can be interpreted as a notion that allows one to compare the complexity of different equivalence relations. Some distinguished equivalence relations are used as benchmarks of complexity. Among these are the relation $=_Y$ of equality for elements of a Polish space Y, and the relation $\simeq_{\mathcal{C}}$ of isomorphism within some class of countable structures $\mathcal{C}$. If E is an equivalence relation on a standard Borel space X, we say that:

- E is *smooth* (or the elements of X are *concretely classifiable* up to E) if E is Borel reducible to $=_Y$ for some Polish space Y;