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Some logically weak Ramseyan theorems

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ABSTRACT

We study four families of consequences of Ramsey's Theorem from the viewpoint of reverse mathematics. The first, which we call the Achromatic Ramsey Theorem, is from a partition relation introduced by Erdős, Hajnal and Rado: $\omega \to [\omega]^r_{c,\leq d}$, which asserts that for every $f : [\omega]^r \to c$ there exists an infinite H with $|f([H]^r)| \leq d$. The second and third are the Free Set Theorem and the Thin Set Theorem, which were introduced by Harvey Friedman. And the last is the Rainbow Ramsey Theorem. We show that, most theorems from these families are quite weak, i.e., they are strictly weaker than ACA_0 over RCA_0. Interestingly, these families turn out to be closely related. We establish the so-called strong cone avoidance property of most instances of the Achromatic Ramsey Theorem by an induction of exponents, then apply this and a similar induction to obtain the strong cone avoidance property of the Free Set Theorem. From the strong cone avoidance property of the Achromatic Ramsey Theorem and the Free Set Theorem, we derive the strong cone avoidance property of the Thin Set Theorem and the Rainbow Ramsey Theorem. It follows easily that a theorem with the strong cone avoidance property does not imply ACA_0 over RCA_0.

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1. Introduction

Reverse mathematics of Ramsey theory has been an active subject for computability theorists for years, in which Ramsey's Theorem for pairs (RT^2_2) has enjoyed being the focus, perhaps since the work of Jockusch [11]. To facilitate the following discussion of Ramsey theory, let us recall some terminology. If X is a set and $0 < r < \omega$, then $[X]^r$ is the set of r-element subsets of X; when we write $[X]^r$, r is always a positive integer. A function f is also called a *coloring* or a *partition*, and its values are naturally called *colors*. A *finite coloring* or *c-coloring* is a function with finite range or with range contained in $c = \{0, 1, \dots, c-1\}$ where c is a positive integer. For a finite coloring $f : [\omega]^r \to c$, a set H is *homogeneous for f* if f is constant on $[H]^r$.

Ramsey's Theorem. *Every $f : [\omega]^r \to c$ for positive c and $r > 1$ admits an infinite f-homogeneous set.*

For a fixed pair r and c, RT^r_c is the instance of Ramsey's Theorem for all c-colorings of r-tuples.

In [11], Jockusch conjectured that some computable two coloring of pairs may have only infinite homogeneous sets computing the halting problem. In the language of reverse mathematics, we may formulate Jockusch's conjecture as: $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \vdash \mathrm{ACA}_0$. This conjecture was later refuted by Seetapun [18]. In his ingenious proof, Seetapun exploited the power of Π^0_1 classes in controlling complexity, which is encapsulated in a theorem of Jockusch and Soare [12]. Seetapun's theorem was later significantly strengthened by Cholak, Jockusch and Slaman [1]. Cholak, Jockusch and Slaman [1] introduced two new ideas which have proven fruitful (e.g., see [10]) and also apply to our work here. The first is the decomposition of RT^2_2 to the so-called COH and SRT^2_2; and the second is replacing a slightly more complicated forcing notion in [18] by Mathias forcing. In [18,1], several questions were raised: whether RT^2_2 implies WKL_0; whether Ramsey's Theorem for stable 2-colorings of pairs (SRT^2_2) is equivalent to RT^2_2; and whether RT^2_2 implies $I\Sigma_2$. Of course, all these questions are based on RCA_0, which is a base theory for most work in reverse mathematics. These had been major open questions in the reverse mathematics of Ramsey theory. The first two have been negatively answered by Jiayi Liu [16] and Chong, Slaman and Yang in [3], respectively. More recently, Chong, Slaman and Yang announced a negative answer to the third question.

Besides these major open questions, various authors have also studied consequences of Ramsey's Theorem, mostly of RT^2_2. Many consequences of RT^2_2 have been shown to be strictly weaker; and relations between these consequences give rise to a complicated picture, which fits the tradition of computability theory quite well. For example, Hirschfeldt and Shore [10] proved that the Ascending and Descending Sequences principle (ADS) and the Chain and Antichain principle (CAC) are both strictly weaker than RT^2_2; Csima and Mileti [5] proved that the Rainbow Ramsey Theorem for pairs (RRT^2_2) does not imply ADS, and thus is also strictly weaker than RT^2_2.