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Traces in monoidal derivators, and homotopy colimits

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ABSTRACT

A variant of the trace in a monoidal category is given in the setting of closed monoidal derivators, which is applicable to endomorphisms of fiberwise dualizable objects. Functoriality of this trace is established. As an application, an explicit formula is deduced for the trace of the homotopy colimit of endomorphisms over finite categories in which all endomorphisms are invertible. This result can be seen as a generalization of the additivity of traces in monoidal categories with a compatible triangulation.

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1. Introduction

1.1. The additivity of traces

Let $\mathcal{C}$ be a symmetric monoidal category which in addition is triangulated. Examples include various "stable homotopy categories" (such as the classical and equivariant in algebraic topology, the motivic in algebraic geometry) or all kinds of "derived categories" (of modules, of perfect complexes on a scheme, etc.). Let X, Y and Z be dualizable objects in $\mathcal{C}$,

$$D : X \to Y \to Z \to^{+}$$

a distinguished triangle, and f an endomorphism of D. The *additivity of traces* is the statement that the following relation holds among the traces of the components of f:

$$\operatorname{tr}(f_Y) = \operatorname{tr}(f_X) + \operatorname{tr}(f_Z). \tag{1}$$

Well-known examples are the additivity of the Euler characteristic of finite CW-complexes ($\chi(Y) = \chi(X) + \chi(Y/X)$ for $X \subset Y$ a subcomplex) or the additivity of traces in short exact sequences of finite dimensional vector spaces. The additivity of traces should be considered as a *principle*: Although incorrect as it stands, it embodies an important idea. One should therefore try to find the right context to formulate this idea precisely and prove it.

In [19], J. Peter May made an important step in this direction. He gave a list of axioms expressing a compatibility of the monoidal and the triangulated structure, and proved that if they are satisfied, then one can always replace f by an endomorphism f' with $f'_X = f_X$ and $f'_Y = f_Y$ such that (1) holds for the components of f'. This result has two drawbacks though: Firstly, there is this awkwardness of f' replacing f, and secondly, the axioms are rather complicated.

As noted in [8], both these drawbacks are related to the well-known deficiencies of triangulated categories. Since the foremost example of a situation in which May's compatibility axioms hold, is when $\mathcal{C}$ is the homotopy category of a stable monoidal model category, it should not come as a surprise that May's result can be reproved in the setting of triangulated derivators. Moreover, since triangulated derivators eliminate some of the problems encountered in triangulated categories, a more satisfying formulation of the additivity of traces should be available. We will describe it now.

Let $\mathbb{D}$ be a closed symmetric monoidal triangulated derivator,[2] and $\ulcorner$ the free category on the following graph:

[2] See Section 2 for the definition of this notion.