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On the string topology category of compact Lie groups

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ABSTRACT

We answer a question of Blumberg, Cohen and Teleman, showing that the Chas–Sullivan loop homology is the Hochschild cohomology of any object in the rational string topology category of a compact, simply connected, Lie group G. Moreover, we show that the answer follows from the classification of the localizing subcategories of the derived category of chains on the based loops of G, which we achieve using the stratification machinery of Benson, Iyengar and Krause. For integral coefficients we get similar results for G a simply-connected special unitary group.

1. Introduction

For a closed, oriented manifold M of dimension m, the *string topology category* St_M of M is a category enriched over chain-complexes over a ground ring k. Its objects are closed, oriented submanifolds of M. The chain-complex of morphisms $\mathrm{Hom}_{\mathsf{St}_M}(N_1, N_2)$ between two objects $N_1, N_2 \in \mathsf{St}_M$ is quasi-isomorphic to the chain-complex of the space $\mathrm{Path}_M(N_1, N_2)$, which is the space of paths from N_1 to N_2 in M. Obviously, for an object

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$N \in \mathsf{St}_M$ the chain-complex of endomorphisms $\mathrm{Hom}_{\mathsf{St}_M}(N,N)$ forms a differential graded algebra (*dga*). Blumberg, Cohen and Teleman defined the string topology category in [4], where they also posed the following question:

Question 1.1. (See [4].) For which connected, closed, oriented submanifolds $N \subset M$ is the Hochschild cohomology of the dga $\mathrm{Hom}_{\mathsf{St}_M}(N,N)$ isomorphic to the homology of the free loop space $\mathrm{L}M$ as algebras?

$$H_{*+m}(\mathrm{L}N;k) \cong HH^* \mathrm{Hom}_{\mathsf{St}_M}(N,N)?$$

The algebra structure on the homology of the free loop space is the *loop product* given by Chas and Sullivan [5].

In fact, for every object $N \in \mathsf{St}_M$ there is a natural map of graded-commutative rings

$$\varphi_N : H_{*+m}(\mathrm{L}M;k) \to HH^*\big(\mathrm{Hom}_{\mathsf{St}_M}(N,N)\big|k\big)$$

whose construction is given in Section 5. So it is natural to ask when is this particular map an isomorphism. The following result provides an answer to this question in several cases.

Theorem 1.2. *Fix a regular commutative ring k and let M be a compact, simply-connected manifold of dimension m satisfying the following conditions:*

1. *$H_*(\Omega M;k)$ is a polynomial ring over k on finitely many generators concentrated in even degrees and*
2. *the natural map $H_{*+m}(\mathrm{L}M) \to H_*(\Omega M)$ is surjective.*

Then for any connected, closed, oriented submanifold $N \subset M$ the map

$$\varphi_N : H_{*+m}(\mathrm{L}M;k) \to HH^*\big(\mathrm{Hom}_{\mathsf{St}_M}(N,N)\big|k\big)$$

is an isomorphism.

The proof of this theorem, and the construction of the machinery and associated results needed for the proof, take most of this paper. At the suggestion of the referee, a guide to the line of argument is provided in Section 2.

Two specific cases in which Theorems 1.2 and 1.4 (see below) hold are given in the next result.

Corollary 1.3. *Theorems 1.2 and 1.4 hold in the following two cases:*

1. *$k = H\mathbb{Q}$ and M is a simply-connected, compact, Lie group;*
2. *$k = H\mathbb{Z}$ and $M = SU(n)$ for $n > 1$.*