

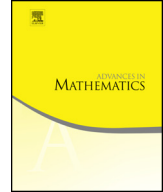


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On the toric ideal of a matroid ☆,☆☆



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ABSTRACT

Describing minimal generating set of a toric ideal is a well-studied and difficult problem. In 1980 White conjectured that the toric ideal associated to a matroid is equal to the ideal generated by quadratic binomials corresponding to symmetric exchanges.

We prove White's conjecture up to saturation, that is that the saturations of both ideals are equal. In the language of algebraic geometry this means that both ideals define the same projective scheme. Additionally we prove the full conjecture for strongly base orderable matroids.

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1. Introduction

Let M be a matroid on a ground set E with the set of bases $\mathfrak{B} \subset \mathcal{P}(E)$ (the reader is referred to [16] for background of matroid theory). For a fixed field $\mathbb{K}$ let $S_M := \mathbb{K}[y_B: B \in \mathfrak{B}]$ be a polynomial ring. Let φ_M be the $\mathbb{K}$ -homomorphism:

$$\varphi_M : S_M \ni y_B \rightarrow \prod_{e \in B} x_e \in \mathbb{K}[x_e: e \in E].$$

The *toric ideal of a matroid* M , denoted by I_M , is the kernel of the map φ_M . For a realizable matroid M the toric variety associated with the ideal I_M has a very nice embedding as a subvariety of a Grassmannian [8]. It is the closure of the torus orbit of the point of the Grassmannian corresponding to the matroid M .

The family $\mathfrak{B}$ of bases, from the definition of a matroid, is nonempty and satisfies *exchange property* — for every bases B_1, B_2 and $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$, such that $(B_1 \setminus e) \cup f$ is also a basis.

Brualdi [3] showed that bases of a matroid satisfy also *symmetric exchange property* — for every bases B_1, B_2 and $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$, such that both $(B_1 \setminus e) \cup f$ and $(B_2 \setminus f) \cup e$ are bases.

Surprisingly, even a stronger property, known as *multiple symmetric exchange property*, is true — for every bases B_1, B_2 and $A_1 \subset B_1$ there exists $A_2 \subset B_2$, such that both $(B_1 \setminus A_1) \cup A_2$ and $(B_2 \setminus A_2) \cup A_1$ are bases (for simple proofs see [14,23], and [12,13] for more exchange properties).

Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is, $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$. Then we say that the quadratic binomial $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$ corresponds to *symmetric exchange*. It is clear that such binomials belong to the ideal I_M . White conjectured that they generate this ideal.

Conjecture 1. (White 1980, [22]) *For every matroid M its toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.*

Since every toric ideal is generated by binomials it is not hard to rephrase the above conjecture in the combinatorial language. It asserts that if two multisets of bases of a matroid have equal union (as a multiset), then one can pass between them by a sequence of symmetric exchanges. In fact this is the original formulation due to White. We immediately see that the conjecture does not depend on the field $\mathbb{K}$.

The most significant partial result is due to Blasiak [1], who confirmed the conjecture for graphical matroids. Kashiwabara [11] checked the case of matroids of rank at most 3. Schweig [18] proved the case of lattice path matroids, which are a subclass of transversal matroids. Recently, Bonin [2] confirmed the conjecture for sparse paving matroids.

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