Advances in Mathematics 259 (2014) 89–115

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Relative Noether inequality on fibered surfaces

Xinyi Yuan [a,*], Tong Zhang [b]

[a] *Department of Mathematics, University of California, Berkeley, CA 94720, USA*
[b] *Department of Mathematics, University of Alberta, Edmonton, AB T6G 2G1, Canada*

ARTICLE INFO

Article history:
Received 27 September 2013
Accepted 18 March 2014
Available online 1 April 2014
Communicated by Ravi Vakil

Keywords:
Noether inequality
Algebraic surface
Fibered surface
Nef line bundle
Linear system
Hilbert–Samuel formula
Slope inequality
Severi inequality

ABSTRACT

We prove effective upper bounds on the global sections of nef line bundles of small generic degree over a fibered surface over a field of any characteristic. It can be viewed as a relative version of the classical Noether inequality for surfaces. As a consequence, we give a new proof of the slope inequality for fibered surface without using any stability method. The treatment is essentially different from those of Xiao, Cornalba–Harris and Moriwaki. We also study the geography problem of surfaces in positive characteristics and show that the Severi inequality is true for surfaces of general type in positive characteristic whose Albanese map is generically finite. Moreover, the geography of surfaces with Albanese fibrations is studied.

Contents

* Corresponding author.
E-mail addresses: yxy@math.berkeley.edu (X. Yuan), tzhang5@ualberta.ca (T. Zhang).

http://dx.doi.org/10.1016/j.aim.2014.03.018
0001-8708/

1. Introduction

This paper is the algebraic version of the previous work [24] of the authors on linear series on arithmetic surfaces.

We prove effective upper bounds on the global sections of nef line bundles of small generic degree over a fibered surface over a field of any characteristic. It can be viewed as a relative version of the classical Noether inequality for surfaces.

As a consequence, we give a new proof of the slope inequality for fibered surface without using any stability method. The treatment is essentially different from those of Xiao, Cornalba–Harris and Moriwaki. We also study the geography problem of surfaces in positive characteristics and show that the Severi inequality is true for surfaces of general type in positive characteristic whose Albanese map is generically finite. Moreover, the geography of surfaces with Albanese fibrations is studied.

We would like to point out that most results in this paper, except the slope inequality, are new in positive characteristic. Nevertheless, we will state our results in full generality, since in characteristic 0 they still hold and Theorems 1.1 and 1.2 have not been stated yet in previous literatures.

1.1. Relative Noether inequality

Let k be an algebraically closed field of any characteristic. Let $f : X \to Y$ be a *surface fibration of genus* g over k. That is:

(1) X is a smooth projective surface over k;
(2) Y is a smooth projective curve over k;
(3) f is flat and the general fiber F of f is a geometrically integral curve of arithmetic genus $g := p_a(F)$.