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On spectral N-Bernoulli measures ☆

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A B S T R A C T

For $0 < \rho < 1$ and $N > 1$ an integer, let μ be the self-similar measure defined by $\mu(\cdot) = \sum_{i=0}^{N-1} \frac{1}{N}\mu(\rho^{-1}(\cdot) - i)$. We prove that $L^2(\mu)$ has an exponential orthonormal basis if and only if $\rho = \frac{1}{q}$ for some $q > 0$ and N divides q. The special case is the Cantor measure with $\rho = \frac{1}{2k}$ and $N = 2$ [16], which was proved recently to be the only spectral measure among the Bernoulli convolutions with $0 < \rho < 1$ [4].

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Contents

1. Introduction

Let μ be a probability measure on $\mathbb{R}^s$ with compact support. For a countable subset $\Lambda \subset \mathbb{R}^s$, we let $e_\Lambda = \{e_\lambda = e^{-2\pi i\langle\lambda,x\rangle}\colon \lambda \in \Lambda\}$. We call μ a spectral measure, and Λ a *spectrum* of μ if e_Λ is an orthogonal basis for $L^2(\mu)$. The existence and nonexistence of a spectrum for μ is a basic problem in harmonic analysis, it was initiated by Fuglede in his seminal paper [13], and has been studied extensively since then [4–10,14,16,19–21, 23,24,27,28]. Recently He, Lai and Lau [14] proved that a spectral measure μ must be of pure type, i.e., μ is absolutely continuous or singular continuous with respect to the Lebesgue measure or counting measure supported on a finite set (actually this holds more generally for *frames*). When μ is the Lebesgue measure restricted on a set K in $\mathbb{R}^s$, it is well-known that the spectral property is closely connected with the tiling property of K, and is known as the Fuglede problem [13,17,19,28]. For continuous singular measures, the first spectral measure was given by Jorgensen and Pedersen [16]: the Cantor measure μ_ρ with contraction ratio $\rho = 1/2k$. There are considerable studies for such measures [5,9,11,15,19,23,24,27], and a celebrated open problem was to characterize the spectral measures μ_ρ, $0 < \rho < 1$ among the Bernoulli convolutions

$$\mu_\rho(\cdot) = 1/2\mu_\rho\big(\rho^{-1}\cdot\big) + 1/2\mu_\rho\big(\rho^{-1}\cdot -1\big).$$

In [15], Hu and Lau showed that μ_ρ admits an infinite orthonormal set if and only if ρ is the n-th root of p/q where p is odd and q is even. The characterization problem was finally completed recently by Dai [4] that the above Cantor measures $\mu_{1/(2k)}$ are the only class of spectral measures among the μ_ρ.

In this paper we study the spectrality of the self-similar measures. Let $0 < \rho < 1$, $\mathcal{D} = \{0, d_1, \ldots, d_{N-1}\}$ be a finite set in $\mathbb{R}$, and $\{w_j\}_{j=0}^{N-1}$ a set of probability weights. We call μ a *self-similar measure* generated by $(\rho, \mathcal{D})$ and $\{w_j\}_{j=0}^{N-1}$ if μ is the unique probability measure satisfying

$$\mu(\cdot) = \frac{1}{N}\sum_{j=0}^{N-1} w_j\mu\big(\rho^{-1}(\cdot) - d_j\big). \tag{1.1}$$

We will use $\mu_{\rho,N}$ to denote the special case where $\mathcal{D} = \{0, \ldots, N-1\}$ with uniform weight, i.e.,