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Isoperimetric type problems and Alexandrov–Fenchel type inequalities in the hyperbolic space

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ABSTRACT

In this paper, we solve various isoperimetric problems for the quermassintegrals and the curvature integrals in the hyperbolic space $\mathbb{H}^n$, by using quermassintegral preserving curvature flows. As a byproduct, we obtain hyperbolic Alexandrov–Fenchel inequalities.

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1. Introduction

Isoperimetric type problems play an important role in mathematics. The classical isoperimetric theorem in the Euclidean space says that among all bounded domains in $\mathbb{R}^n$ with given volume, the minimum of the area of the boundary is achieved precisely by the round balls. This can be formulated as an optimal inequality

$$\operatorname{Area}(\partial K) \geqslant n^{\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}} \operatorname{Vol}(K)^{\frac{n-1}{n}}, \tag{1.1}$$

for any bounded domain $K \subset \mathbb{R}^n$, and equality holds if and only if K is a geodesic ball. Here and throughout this paper, ω_k denotes the k-th dimensional Hausdorff measure of the k-dimensional unit sphere $\mathbb{S}^k$, and by a bounded domain we mean a compact set with non-empty interior. When $n = 2$, inequality (1.1) is

$$L^2 \geqslant 4\pi A, \tag{1.2}$$

where L is the length of a closed curve γ in $\mathbb{R}^2$ and A is the area of the enclosed domain by γ. Inequalities (1.1) and (1.2) are the classical isoperimetric inequalities. Their general forms are the Alexandrov–Fenchel quermassintegral inequalities. A special, but interesting class of the Alexandrov–Fenchel quermassintegral establishes the relationship between the quermassintegrals or the curvature integrals:

$$\int_{\partial K} H_k \, d\mu \geqslant \omega_{n-1}^{\frac{k-l}{n-1-l}} \left(\int_{\partial K} H_l \, d\mu \right)^{\frac{n-1-k}{n-1-l}}, \quad 0 \leqslant l < k \leqslant n-1, \tag{1.3}$$

for any convex bounded domain $K \subset \mathbb{R}^n$ with C^2 boundary, where H_k is the *(normalized) k-th mean curvature* of ∂K as an embedding in $\mathbb{R}^n$. These inequalities have been intensively studied by many mathematicians and have many applications in differential geometry and integral geometry. See the excellent books of Burago–Zalgaller [7], Santalo [37] and Schneider [39]. Recently, the Alexandrov–Fenchel quermassintegral inequalities in $\mathbb{R}^n$ have been extended to certain classes of non-convex domains. See for example [11,25,29].

All these above inequalities solve the problem if one geometric quantity attains its minimum or maximum at geodesic balls among a class of (smooth) bounded domains in $\mathbb{R}^n$ with another given geometric quantity. We call such problems *isoperimetric type problems*.

It is a very natural question to ask if such isoperimetric type problems also hold in the hyperbolic space $\mathbb{H}^n$. We remark that in this paper $\mathbb{H}^n$ denotes the hyperbolic space with the sectional curvature -1. One of the main motivations to study this problem comes naturally from integral geometry in $\mathbb{H}^n$. Another main motivation comes from the recent study of ADM mass, Gauss–Bonnet–Chern mass and quasi-local mass in asymptotically hyperbolic manifolds, see [22]. The isoperimetric problem between volume and area in $\mathbb{H}^n$