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A universal formula for deformation quantization on Kähler manifolds

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ABSTRACT

We give an explicit local formula for any formal deformation quantization, with separation of variables, on a Kähler manifold. The formula is given in terms of differential operators, parametrized by acyclic combinatorial graphs.

1. Introduction

Among the first to systematically develop the notion of deformation quantization were Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer. In [1] and [2], they developed the notion of quantization as a deformation of the commutative algebra of classical observables through a family of non-commutative products $\star_h$, parametrized by a real parameter h, and gave an independent formulation of quantum mechanics using this notion.

As opposed to other approaches to quantization, such as geometric quantization, the theory of deformation quantization does not attempt to construct a space of quantum states, but focuses on the algebraic structure of the space of observables.

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Much work has been done on the theory of deformation quantization, and its formal counterpart, where h is interpreted as a formal parameter. In its most general context, deformation quantization is studied on Poisson manifolds. In [8], Kontsevich proves the existence of a formal deformation quantization on any Poisson manifold. Moreover, he gives a formula for a deformation quantization of any Poisson structure on $\mathbb{R}^n$. His formula describes the star product in terms of bidifferential operators parametrized by graphs and with coefficients given by integrals over appropriate configuration spaces. This bears resemblance in flavour to the construction presented in this paper, which is also based on a certain interpretation of graphs as differential operators.

Other significant constructions of star products include the geometrical construction by Fedosov in [5], where he constructs a deformation quantization on an arbitrary symplectic manifold. Moreover, we should mention the work of Schlichenmaier [10], where he uses the theory of Toeplitz operators to construct a deformation quantization on any compact Kähler manifold.

The question of existence and classification of deformation quantizations on an arbitrary symplectic manifold was solved by De Wilde and Lecomte in [4], where they show that equivalence classes of star products are classified by formal cohomology classes. On Kähler manifolds, existence and classification was addressed by Karabegov in [6], where he proves that deformation quantizations with separation of variables are classified, completely and not only up to equivalence, by closed formal $(1,1)$-forms, which he calls formal deformations of the Kähler form. In this paper, we shall be dealing exclusively with deformation quantizations, with separation of variables, on Kähler manifolds.

In this setting, Berezin [3] originally wrote down integral formulas for a star product, but he had to make severe assumptions on the Kähler manifold. By interpreting Berezin's integral formulas formally, and studying their asymptotic behaviour, Reshetikhin and Takhtajan [9] gave an explicit formula, in terms of Feynman graphs, for a formal deformation quantization on any Kähler manifold.

Reshetikhin and Takhtajan applied the method of stationary phase to Berezin's integrals to obtain the asymptotic expansion, and the description in terms of Feynman graphs arises in a natural way through this approach. However, the graphs produced by the expansion of Berezin's integrals have relations among them, expressing fundamental identities on the Kähler manifold. Moreover, the expansion produces disconnected graphs which prevent the star product from being normalized.

Using the general existence of a unit, Reshetikhin and Takhtajan defined a normalized version of the star product. The coefficients of the unit for the non-normalized star product can be determined inductively by solving the defining equations for the unit, but this approach does not yield an explicit formula for the unit in terms of Feynman graphs, and consequently such a formula for the normalized star product was not given.

The present paper grew out of an attempt to find an explicit formula for this normalized star product of Reshetikhin and Takhtajan in terms of graphs. The crucial observation is that relations among the graphs, as well as the fact that the star product is not normalized, are caused by graphs with cycles.