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Functional integral approach to semi-relativistic Pauli–Fierz models

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ABSTRACT

By means of functional integrations spectral properties of semi-relativistic Pauli–Fierz Hamiltonians

$$H = \sqrt{(\mathrm{p} - \alpha \mathrm{A})^2 + m^2} - m + V + \mathrm{H}_{\mathrm{rad}}$$

in quantum electrodynamics are considered. Here p is the momentum operator, A a quantized radiation field on which an ultraviolet cutoff is imposed, V an external potential, $\mathrm{H}_{\mathrm{rad}}$ the free field Hamiltonian and $m \geqslant 0$ describes the mass of electron. Two self-adjoint extensions of a semi-relativistic Pauli–Fierz Hamiltonian are defined. The Feynman–Kac type formula of e^{-tH} is given. A self-adjointness, a spatial decay of bound states, a Gaussian domination of the ground state and the existence of a measure associated with the ground state are shown. All the results are independent of values of coupling constant α, and it is emphasized that $m = 0$ is included.

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1. Introduction

1.1. Preliminary

In the past decade a great deal of work has been devoted to studying spectral properties of non-relativistic quantum electrodynamics in the purely mathematical point of view. In this paper we are concerned with the semi-relativistic Pauli–Fierz model (it is abbreviated as SRPF model) in quantum electrodynamics and its spectral properties by using functional integrations. The SRPF model describes a minimal interaction between semi-relativistic electrons and a massless quantized radiation field A on which an ultraviolet cutoff function is imposed. We assume throughout this paper that the electron is spinless and moves in d ($\geqslant 3$) dimensional Euclidean space for simplicity. In the case where the electron has spin 1/2, the procedure is similar and we shall publish details somewhere. A Hamiltonian of semi-relativistic as well as non-relativistic quantum electrodynamics is usually described as a self-adjoint operator in the tensor product of a Hilbert space and a boson Fock space. In this paper instead of the boson Fock space we can formulate the Hamiltonian as a self-adjoint operator in the known Schrödinger representation in a functional realization of the boson Fock space as a space of square integrable functions with respect to the corresponding Gaussian measure. Through the Schrödinger representation a Fyenman–Kac type formula of the strongly continuous one parameter semigroup generated by the SRPF Hamiltonian is given. A functional integral or a path measure approach is proven to be useful to study properties of bound states associated with embedded eigenvalues in the continuous spectrum. See e.g., [22, Sections 6 and 7]. We are interested in investigating properties of bound states and ground states of the SRPF Hamiltonian by functional integrations.

1.2. Self-adjoint extensions and functional integrations

The SRPF Hamiltonian can be realized as a self-adjoint operator bounded from below in the tensor product of $L^2(\mathbb{R}^d)$ and a boson Fock space $\mathscr{F}$, where $L^2(\mathbb{R}^d)$ denotes the state space of a semi-relativistic electron and $\mathscr{F}$ that of photons. Then the decoupled Hamiltonian is given by

$$\left(\sqrt{\mathrm{p}^2+m^2}-m+V\right)\otimes \mathbb{1}+\mathbb{1}\otimes \mathrm{H}_{\mathrm{rad}}, \tag{1.1}$$

where $\mathrm{p}=(\mathrm{p}_1,\ldots,\mathrm{p}_d)=(-i\partial_{x_1},\ldots,-i\partial_{x_d})$ denotes the momentum operator, m electron mass, $V:\mathbb{R}^d\to\mathbb{R}$ an external potential, and $\mathrm{H}_{\mathrm{rad}}$ the free field Hamiltonian on $\mathscr{F}$. The SRPF Hamiltonian is defined by introducing the minimal coupling by the quantized radiation field A with cutoff function $\hat{\varphi}$, i.e., replacing $\mathrm{p}\otimes\mathbb{1}$ with $\mathrm{p}\otimes\mathbb{1}-\alpha A$ and, then

$$\mathrm{H}=\sqrt{(\mathrm{p}\otimes\mathbb{1}-\alpha A)^2+m^2}-m+V\otimes\mathbb{1}+\mathbb{1}\otimes\mathrm{H}_{\mathrm{rad}}, \tag{1.2}$$