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A characterization of saturated fusion systems over abelian 2-groups

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ABSTRACT

Given a saturated fusion system $\mathcal{F}$ over a 2-group S, we prove that S is abelian provided any element of S is $\mathcal{F}$-conjugate to an element of $Z(S)$. This generalizes a Theorem of Camina–Herzog, leading to a significant simplification of its proof. More importantly, it follows that any 2-block B of a finite group has abelian defect groups if all B-subsections are major. Furthermore, every 2-block with a symmetric stable center has abelian defect groups.

1. Introduction

This short note gives an example of how a conjecture in the modular representation theory of finite groups can be proved by showing its generalization to saturated fusion systems. We refer the reader to [1] for definitions and basic results regarding fusion systems and to [6] as a background reference on block theory. Here is our main theorem:

Theorem 1. *Let $\mathcal{F}$ be a saturated fusion system on a 2-group S such that for any $x \in S$, $\mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, Z(S)) \neq \emptyset$. Then S is abelian.*

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Since for any finite group G with Sylow 2-subgroup S the fusion system $\mathcal{F}_S(G)$ is saturated, the above theorem yields immediately the following corollary:

Corollary 1 *(Camina–Herzog). Let G be a finite group such that $|G : C_G(x)|$ is odd for any 2-element x of G. Then the Sylow 2-subgroups of G are abelian.*

Corollary 1 was first proved by Camina and Herzog [2, Theorem 6]. As they point out, it means that one can read from the character table of a finite group if its Sylow 2-subgroups are abelian. The proof of Camina–Herzog relies on the Theorem of Goldschmidt [3] about groups with strongly closed abelian 2-subgroups, whereas our approach is elementary and self-contained. More precisely, we prove Theorem 1 by an induction argument which appears natural in the context of fusion systems. Using the same idea, one can also give an elementary direct proof of Corollary 1 which does not use fusion systems; see Remark 2.1 for details.

We now turn attention to 2-blocks of finite groups, where by a p-block we mean an indecomposable direct summand of the group algebra kG as a (kG, kG)-bimodule for an algebraically closed field k of characteristic $p \neq 0$. The Brauer categories of p-blocks, which we introduce in Definition 2.2, provide important examples of saturated fusion systems. Given a finite group G and a p-block B of G, recall that a *B-subsection* of G is a pair (x, b) such that x is a p-element and b is a block of $C_G(x)$ with the property that the induced block b^G equals B. A B-subsection (x, b) is called *major* if the defect groups of b are also defect groups of B. Theorem 1 applied to the Brauer category of a 2-block yields the following corollary, which we prove in detail at the end of this paper:

Corollary 2. *Suppose B is a 2-block of a finite group G such that all B-subsections are major. Then the defect groups of B are abelian.*

Neither Theorem 1 nor Corollary 1 or Corollary 2 have obvious generalizations replacing the prime 2 by an arbitrary prime p. This is because for $p \in \{3, 5\}$ there are finite groups G in which the centralizer of any p-element has index prime to p and the Sylow p-subgroups of G are extraspecial of order p^3 and exponent p; see the examples listed in [5, Theorem A]. The corresponding statement of Theorem 1 fails also at the prime 7, as the exotic 7-fusion systems discovered by Ruiz and Viruel [7] show.

We conclude with a further application of Theorem 1, concerning 2-blocks of finite groups with a symmetric stable center. Recall the definitions of a symmetric algebra and the stable center of an algebra from [4]. Given a p-block B of a finite group G and a maximal B subpair (P, e), Kessar and Linckelmann [4, Theorem 1.2] prove that the stable center of B is symmetric provided the defect group P of B is abelian and $N_G(P, e)/C_G(P)$ acts freely on $P\backslash\{1\}$. Furthermore, by [4, Theorem 1.1], the converse implication is true if B is the principal block of G. Our next corollary can be seen as a partial answer to the question in how far the converse holds for arbitrary p-blocks. This result is a direct consequence of Theorem 1, [4, Theorem 3.1] and the fact that the Brauer category of a p-block is a saturated fusion system.