Advances in Mathematics 257 (2014) 266–284

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

On the classification of projectively flat Finsler metrics with constant flag curvature

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ARTICLE INFO

Article history:
Received 12 August 2013
Accepted 19 February 2014
Available online 12 March 2014
Communicated by Erwin Lutwak

Keywords:
Finsler metric
Projectively flat
Constant flag curvature

ABSTRACT

In this paper, we study locally projectively flat Finsler metrics with constant flag curvature **K**. We prove those are totally determined by their behaviors at the origin by solving some nonlinear PDEs. The classifications when $\mathbf{K} = 0$, $\mathbf{K} = -1$ and $\mathbf{K} = 1$ are given respectively in an algebraic way. Further, we construct a new projectively flat Finsler metric with flag curvature $\mathbf{K} = 1$ determined by a Minkowski norm with double square roots at the origin. As an application of our main theorems, we give the classification of locally projectively flat spherical symmetric Finsler metrics much easier than before.

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1. Introduction

The regular case of Hilbert's Fourth Problem is to study and characterize Finsler metrics on an open subset in R^n whose geodesics are straight lines. Such metrics are called *locally projectively flat* Finsler metrics. Riemannian metrics form a special and important class in Finsler geometry. Beltrami's theorem tells us that a Riemannian

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[1] Research is supported by the NNSFC (11371209), ZPNSFC (LY13A010013), Scientific Research Fund of Zhejiang Provincial Education Department (Y201326630), Ningbo Natural Science Foundation (2013A610101) and K.C. Wong Magna Fund in Ningbo University.

http://dx.doi.org/10.1016/j.aim.2014.02.022
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metric is locally projectively flat if and only if it is with constant sectional curvature $\mathbf{K} = \lambda$, which can be expressed as

$$F_\lambda = \frac{\sqrt{|y|^2 + \lambda(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \lambda|x|^2}, \tag{1.1}$$

where $y \in T_x\mathcal{U} \approx R^n$, $\mathcal{U} \subset R^n$. However, it is not true in general.

Flag curvature is an analogue of sectional curvature in Finsler geometry. It is known that there are many locally projectively flat Finsler metrics which are not of constant flag curvature; and there are many Finsler metrics with constant flag curvature which are not locally projectively flat. A natural problem is to characterize projectively flat Finsler metrics with constant flag curvature. In [5,6], P. Funk classified projectively flat Finsler metrics with constant flag curvature on convex domains in R^2. The famous Funk metric $F = F(x, y)$ defined on unit ball B^n in R^n is locally projectively flat with flag curvature $\mathbf{K} = -\frac{1}{4}$. It is given by

$$F = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2}, \tag{1.2}$$

where $y \in T_x\mathrm{B}^n \approx R^n$. In 1929, L. Berwald studied locally projectively flat Finsler metrics, specially in the case of zero flag curvature [1,2]. He gave the equivalent equations of such metrics and found that the key problem is to solve the following PDE:

$$\Phi_{x^k} = \Phi\Phi_{y^k}, \tag{1.3}$$

where $\Phi = \Phi(x, y)$, x, $y \in R^n$. However, it is difficult to solve the above equation at that time though he constructed a projectively flat Finsler metric with $\mathbf{K} = 0$ which be called Berwald's metric now as follows

$$B = \frac{(\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle)^2}{(1 - |x|^2)^2\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}, \tag{1.4}$$

where $y \in T_x\mathrm{B}^n \approx R^n$. The first locally projectively flat non-Riemannian Finsler metric with positive flag curvature $\mathbf{K} = 1$ was given by R. Bryant on S^2 [3,4]. By algebraic equations, Z. Shen gave the following expression of Bryant's example including the higher dimension in [10]:

$$\begin{aligned} F(x, y) &= \mathcal{I}m\left[\frac{-\langle x, y\rangle + i\sqrt{(e^{2i\alpha} + |x|^2)|y|^2 - \langle x, y\rangle^2}}{e^{2i\alpha} + |x|^2}\right] \\ &= \sqrt{\frac{\sqrt{\mathcal{A}} + \mathcal{B}}{2\mathcal{D}} + \left(\frac{\mathcal{C}}{\mathcal{D}}\right)^2} + \frac{\mathcal{C}}{\mathcal{D}}, \end{aligned} \tag{1.5}$$

where

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