

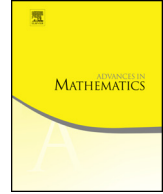


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# The fractal dimensions of the spectrum of Sturm Hamiltonian

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## ABSTRACT

Let  $\alpha \in (0, 1)$  be irrational and  $[0; a_1, a_2, \dots]$  be the continued fraction expansion of  $\alpha$ . Let  $H_{\alpha, V}$  be the Sturm Hamiltonian with frequency  $\alpha$  and coupling  $V$ ,  $\Sigma_{\alpha, V}$  be the spectrum of  $H_{\alpha, V}$ . The fractal dimensions of the spectrum have been determined by Fan, Liu and Wen (2011) [8] when  $\{a_n\}_{n \geq 1}$  is bounded. The present paper will treat the most difficult case, i.e.,  $\{a_n\}_{n \geq 1}$  is unbounded. We prove that for  $V \geq 24$ ,

$$\dim_H \Sigma_{\alpha, V} = s_*(V) \quad \text{and} \quad \overline{\dim}_B \Sigma_{\alpha, V} = s^*(V),$$

where  $s_*(V)$  and  $s^*(V)$  are lower and upper pre-dimensions respectively. By this result, we determine the fractal dimensions of the spectrums for all Sturm Hamiltonians.

We also show the following results:  $s_*(V)$  and  $s^*(V)$  are Lipschitz continuous on any bounded interval of  $[24, \infty)$ ; the limits  $s_*(V) \ln V$  and  $s^*(V) \ln V$  exist as  $V$  tends to infinity, and the limits are constants only depending on  $\alpha$ ;  $s^*(V) = 1$  if and only if  $\limsup_{n \rightarrow \infty} (a_1 \cdots a_n)^{1/n} = \infty$ , which can be compared with the fact:  $s_*(V) = 1$  if and only if  $\liminf_{n \rightarrow \infty} (a_1 \cdots a_n)^{1/n} = \infty$  (Liu and Wen, 2004) [13].

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**1. Introduction**

The Sturm Hamiltonian is a discrete Schrödinger operator

$$(H\psi)_n := \psi_{n-1} + \psi_{n+1} + v_n\psi_n$$

on  $\ell^2(\mathbb{Z})$ , where the potential  $(v_n)_{n \in \mathbb{Z}}$  is given by

$$v_n = V\chi_{[1-\alpha,1)}(n\alpha + \phi \bmod 1), \quad \forall n \in \mathbb{Z}, \tag{1}$$

where  $\alpha \in (0, 1)$  is irrational, and is called frequency,  $V > 0$  is called coupling,  $\phi \in [0, 1)$  is called phase. It is known that the spectrum of Sturm Hamiltonian is independent of  $\phi$ , so we take  $\phi = 0$  and denote the spectrum by  $\Sigma_{\alpha,V}$ . We often simplify the notation  $\Sigma_{\alpha,V}$  to  $\Sigma_V$  or  $\Sigma$  when  $\alpha$  or  $V$  are fixed. The present paper is devoted to determine the fractal dimensions of  $\Sigma_{\alpha,V}$  for all irrational  $\alpha$ .

The most prominent model among the Sturm Hamiltonian is the Fibonacci Hamiltonian, which is given by taking  $\alpha$  to be the golden number  $\alpha_0 := (\sqrt{5} - 1)/2$ . This model was introduced by physicists to model the quasicrystal system [11,16]. Sütö showed that the spectrum always has zero Lebesgue measure [18],

$$L(\Sigma_{\alpha_0,V}) = 0, \quad \text{for all } V > 0.$$

Then it is natural to ask what is the fractal dimension of the spectrum. Raymond first estimated the Hausdorff dimension [17], and he showed that  $\dim_H \Sigma_{\alpha_0,V} < 1$  for  $V > 4$ . Jitomirskaya and Last [10] showed that for any  $V > 0$ , the spectral measure of the operator has positive Hausdorff dimension, as a consequence  $\dim_H \Sigma_{\alpha_0,V} > 0$ . By using dynamical method, Damanik et al. [3] showed that if  $V \geq 16$  then

$$\dim_B \Sigma_{\alpha_0,V} = \dim_H \Sigma_{\alpha_0,V}. \tag{2}$$

They also got lower and upper bounds for the dimensions. Due to these bounds they further showed that

$$\lim_{V \rightarrow \infty} \dim_H \Sigma_{\alpha_0,V} \ln V = \ln(1 + \sqrt{2}). \tag{3}$$

We remark that more than a natural question, the fractal dimensions of the spectrum are also related to the rates of propagation of the fastest part of the wavepacket (see [3] for detail).

Write  $d(V) = \dim_H \Sigma_{\alpha_0,V}$ . Cantat [2], Damanik and Gorodetski [4] showed that:  $d(V) \in (0, 1)$  is analytic on  $(0, \infty)$ . In [5], Damanik and Gorodetski further showed that  $\lim_{V \downarrow 0} d(V) = 1$  and the speed is linear.

Now we go back to the general Sturm Hamiltonian case. We fix an irrational  $\alpha \in (0, 1)$  with continued fraction expansion  $[0; a_1, a_2, \dots]$ . Write

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