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The fractal dimensions of the spectrum of Sturm Hamiltonian

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ABSTRACT

Let $\alpha \in (0,1)$ be irrational and $[0; a_1, a_2, \ldots]$ be the continued fraction expansion of α. Let $H_{\alpha,V}$ be the Sturm Hamiltonian with frequency α and coupling V, $\Sigma_{\alpha,V}$ be the spectrum of $H_{\alpha,V}$. The fractal dimensions of the spectrum have been determined by Fan, Liu and Wen (2011) [8] when $\{a_n\}_{n\geqslant 1}$ is bounded. The present paper will treat the most difficult case, i.e., $\{a_n\}_{n\geqslant 1}$ is unbounded. We prove that for $V \geqslant 24$,

$$\dim_H \Sigma_{\alpha,V} = s_*(V) \quad \text{and} \quad \overline{\dim}_B \Sigma_{\alpha,V} = s^*(V),$$

where $s_*(V)$ and $s^*(V)$ are lower and upper pre-dimensions respectively. By this result, we determine the fractal dimensions of the spectrums for all Sturm Hamiltonians.

We also show the following results: $s_*(V)$ and $s^*(V)$ are Lipschitz continuous on any bounded interval of $[24,\infty)$; the limits $s_*(V)\ln V$ and $s^*(V)\ln V$ exist as V tends to infinity, and the limits are constants only depending on α; $s^*(V)=1$ if and only if $\limsup_{n\to\infty}(a_1\cdots a_n)^{1/n}=\infty$, which can be compared with the fact: $s_*(V)=1$ if and only if $\liminf_{n\to\infty}(a_1\cdots a_n)^{1/n}=\infty$ (Liu and Wen, 2004) [13].

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1. Introduction

The Sturm Hamiltonian is a discrete Schrödinger operator

$$(H\psi)_n := \psi_{n-1} + \psi_{n+1} + v_n\psi_n$$

on $\ell^2(\mathbb{Z})$, where the potential $(v_n)_{n\in\mathbb{Z}}$ is given by

$$v_n = V\chi_{[1-\alpha,1)}(n\alpha + \phi \bmod 1), \quad \forall n \in \mathbb{Z}, \tag{1}$$

where $\alpha \in (0,1)$ is irrational, and is called frequency, $V > 0$ is called coupling, $\phi \in [0,1)$ is called phase. It is known that the spectrum of Sturm Hamiltonian is independent of ϕ, so we take $\phi = 0$ and denote the spectrum by $\Sigma_{\alpha,V}$. We often simplify the notation $\Sigma_{\alpha,V}$ to Σ_V or Σ when α or V are fixed. The present paper is devoted to determine the fractal dimensions of $\Sigma_{\alpha,V}$ for all irrational α.

The most prominent model among the Sturm Hamiltonian is the Fibonacci Hamiltonian, which is given by taking α to be the golden number $\alpha_0 := (\sqrt{5}-1)/2$. This model was introduced by physicists to model the quasicrystal system [11,16]. Sütö showed that the spectrum always has zero Lebesgue measure [18],

$$L(\Sigma_{\alpha_0,V}) = 0, \quad \text{for all } V > 0.$$

Then it is natural to ask what is the fractal dimension of the spectrum. Raymond first estimated the Hausdorff dimension [17], and he showed that $\dim_H \Sigma_{\alpha_0,V} < 1$ for $V > 4$. Jitomirskaya and Last [10] showed that for any $V > 0$, the spectral measure of the operator has positive Hausdorff dimension, as a consequence $\dim_H \Sigma_{\alpha_0,V} > 0$. By using dynamical method, Damanik et al. [3] showed that if $V \geqslant 16$ then

$$\dim_B \Sigma_{\alpha_0,V} = \dim_H \Sigma_{\alpha_0,V}. \tag{2}$$

They also got lower and upper bounds for the dimensions. Due to these bounds they further showed that

$$\lim_{V\to\infty} \dim_H \Sigma_{\alpha_0,V} \ln V = \ln(1+\sqrt{2}\,). \tag{3}$$

We remark that more than a natural question, the fractal dimensions of the spectrum are also related to the rates of propagation of the fastest part of the wavepacket (see [3] for detail).

Write $d(V) = \dim_H \Sigma_{\alpha_0,V}$. Cantat [2], Damanik and Gorodetski [4] showed that: $d(V) \in (0,1)$ is analytic on $(0,\infty)$. In [5], Damanik and Gorodetski further showed that $\lim_{V\downarrow 0} d(V) = 1$ and the speed is linear.

Now we go back to the general Sturm Hamiltonian case. We fix an irrational $\alpha \in (0,1)$ with continued fraction expansion $[0; a_1, a_2, \ldots]$. Write