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# Abelian varieties in Brill–Noether loci [☆]

Ciro Ciliberto [a], Margarida Mendes Lopes [b,*], Rita Pardini [c]

[a] *Dipartimento di Matematica, II Università di Roma, Italy*
[b] *Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*
[c] *Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo, 5, 56127 Pisa, Italy*

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ABSTRACT

In this paper, improving on results of Abramovich, Harris, Debarre and Fahlaoui [1,8], we give the full classification of curves $C$ of genus $g$ such that a Brill–Noether locus $W^s_d(C)$, strictly contained in the jacobian $J(C)$ of $C$, contains a variety $Z$ stable under translations by the elements of a positive dimensional abelian subvariety $A \subsetneq J(C)$ and such that $\dim(Z) = d - \dim(A) - 2s$, i.e., the maximum possible for such a $Z$.



**Contents**



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* Corresponding author.
*E-mail addresses:* cilibert@axp.mat.uniroma2.it (C. Ciliberto), mmlopes@math.tecnico.ulisboa.pt (M.M. Lopes), pardini@dm.unipi.it (R. Pardini).

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## 1. Introduction

In [1] the authors posed the problem of studying, and possibly classifying, situations like this:

(∗) $C$ is a smooth, projective, complex curve of genus $g$, $Z$ is an irreducible $r$-dimensional subvariety of a Brill–Noether locus $W^s_d(C) \subsetneq J^d(C)$, and $Z$ is stable under translations by the elements of an abelian subvariety $A \subsetneq J(C)$ of dimension $a > 0$ (if so, we will say that $Z$ is $A$-*stable*).

Actually in [1] the variety $Z$ *is* the translate of a positive dimensional proper abelian subvariety of $J(C)$, while the above slightly more general formulation was given in [8].

The motivation for studying (∗) resides, among other things, in a theorem of Faltings (see [9]) to the effect that if $X$ is an abelian variety defined over a number field $\mathbb{K}$, and $Z \subsetneq X$ is a subvariety not containing any translate of a positive dimensional abelian subvariety of $X$, then the number of rational points of $Z$ over $\mathbb{K}$ is finite. The idea in [1] was to apply Faltings' theorem to the $d$-fold symmetric product $C(d)$ of a curve $C$ defined over a number field $\mathbb{K}$. If $C$ has no positive dimensional linear series of degree $d$, then $C(d)$ is isomorphic to its Abel–Jacobi image $W_d(C)$ in $J^d(C)$. Thus $C(d)$ has finitely many rational points over $\mathbb{K}$ if $W_d(C)$ does not contain any translate of a positive dimensional abelian subvariety of $J(C)$. The suggestion in [1] is that, if, by contrast, $W_d(C)$ contains the translate of a positive dimensional abelian subvariety of $J(C)$, then $C$ should be *quite special*, e.g., it should admit a map to a curve of lower positive genus (curves of this kind clearly are in situation (∗)). This idea was tested in [1], where a number of partial results were proven for low values of $d$.

The problem was taken up in [8], see also [7], where, among other things, it is proven that if (∗) holds, then $r + a + 2s \leqslant d$, and, if in addition $d + r \leqslant g - 1$, then $r + a + 2s = d$ if and only if:

(a) there is a degree 2 morphism $\varphi : C \to C'$, with $C'$ a smooth curve of genus $a$, such that $A = \varphi^*(J(C'))$ and $Z = W_{d-2a-2s}(C) + \varphi^*(J^{a+s}(C'))$.

In [8] there is also the following example with $(d, s) = (g - 1, 0)$:

(b) there is an (étale) degree 2 morphism $\varphi : C \to C'$, with $C'$ a smooth curve of genus $g' = r + 1$, $A$ is the Prym variety of $\varphi$ and $Z \subset W_{g-1}(C)$ is the connected component of $\varphi_*^{-1}(K_{C'})$ consisting of divisor classes $D$ with $h^0(\mathcal{O}_C(D))$ odd, where $\varphi_* : J^{g-1}(C) \to J^{g-1}(C')$ is the *norm map*. One has $Z \cong A$, hence $r = a$.