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The possible values of critical points between strongly congruence-proper varieties of algebras ☆

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ABSTRACT

We denote by $\operatorname{Con_c} A$ the $(\vee, 0)$-semilattice of all finitely generated congruences of an (universal) algebra A, and we define $\operatorname{Con_c} \mathcal{V}$ as the class of all isomorphic copies of all $\operatorname{Con_c} A$, for $A \in \mathcal{V}$, for any variety $\mathcal{V}$ of algebras.

Let $\mathcal{V}$ and $\mathcal{W}$ be locally finite varieties of algebras such that for each finite algebra $A \in \mathcal{V}$ there are, up to isomorphism, only finitely many $B \in \mathcal{W}$ such that $\operatorname{Con_c} A \cong \operatorname{Con_c} B$, and every such B is finite. If $\operatorname{Con_c} \mathcal{V} \nsubseteq \operatorname{Con_c} \mathcal{W}$, then there exists a $(\vee, 0)$-semilattice of cardinality $\aleph_2$ in $(\operatorname{Con_c} \mathcal{V}) - (\operatorname{Con_c} \mathcal{W})$. Our result extends to quasivarieties of first-order structures, with finitely many relation symbols, and relative congruence lattices.

In particular, if $\mathcal{W}$ is a finitely generated variety of algebras, then this occurs in case $\mathcal{W}$ omits the tame congruence theory types **1** and **5**; which, in turn, occurs in case $\mathcal{W}$ satisfies a nontrivial congruence identity.

The bound $\aleph_2$ is sharp.

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1. Introduction

Why do so many representation problems in algebra, enjoying positive solutions in the finite case, have counterexamples of minimal cardinality either $\aleph_0$, $\aleph_1$, or $\aleph_2$, and no other cardinality? By a *representation problem*, we mean that we are given *categories* $\mathcal{A}$ and $\mathcal{B}$ together with a *functor* $\Phi : \mathcal{A} \to \mathcal{B}$, and we are trying to determine whether an object B of $\mathcal{B}$ is isomorphic to $\Phi(A)$ for some object A of $\mathcal{A}$. We are also given a mapping from the objects of $\mathcal{B}$ to the cardinals, that behaves like the cardinality mapping on sets.

Examples of such representation problems cover various fields of mathematics. Here are a few examples, among many:

- Every (at most) countable Boolean algebra is generated by a chain (cf. [9, Theorem 172]), but not every Boolean algebra is generated by a chain (cf. [9, Lemma 179]). It is an easy exercise to verify that in fact, every subchain C of the free Boolean algebra F on $\aleph_1$ generators is countable, thus F cannot be generated by C.
- Every dimension group with at most $\aleph_1$ elements is isomorphic to $K_0(R)$ for some (von Neumann) regular ring R (cf. [1,7]), but there is a dimension group with $\aleph_2$ elements which is not isomorphic to $K_0(R)$ for any regular ring R (cf. [23]).
- Every distributive algebraic lattice with at most $\aleph_1$ compact elements is isomorphic to the congruence lattice of some lattice (cf. [11–13]), but not every distributive algebraic lattice is isomorphic to the congruence lattice of some lattice (cf. [24]); the minimal number of compact elements in a counterexample, namely $\aleph_2$, is obtained in [20].

In an earlier paper [3], we introduced a particular case of the kind of representation problem considered above, concentrated in the notion of *critical point* between two varieties of (universal) algebras. It turned out that this notion often behaves as a paradigm for those kinds of problems. The present paper will be centered on that paradigm, and will offer an explanation, in that context, why for so many representation problems, the minimal size of a counterexample (if it exists at all) lies below $\aleph_2$. Although initially stated for universal algebras, the method of proof of our main result (Theorem 5.1) carries a potential of generalization to many other contexts, starting with Theorem 6.1.

Let us be a bit more precise. For an algebra A we denote by $\operatorname{Con_c} A$ the $(\vee, 0)$-semilattice of all compact (i.e., finitely generated) congruences of A. A *lifting* of a $(\vee, 0)$-semilattice S is an algebra A such that $\operatorname{Con_c} A \cong S$. For a variety $\mathcal{V}$ of algebras we denote by $\operatorname{Con_c} \mathcal{V}$ the class of all $(\vee, 0)$-semilattices with a lifting in $\mathcal{V}$.

For varieties $\mathcal{V}$ and $\mathcal{W}$ of algebras, the *critical point* between $\mathcal{V}$ and $\mathcal{W}$, denoted by $\operatorname{crit}(\mathcal{V}; \mathcal{W})$, is the smallest cardinality of a member of $(\operatorname{Con_c} \mathcal{V}) - (\operatorname{Con_c} \mathcal{W})$ if $\operatorname{Con_c} \mathcal{V} \nsubseteq \operatorname{Con_c} \mathcal{W}$, and ∞ otherwise (cf. [3,22]).