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The isoperimetrix in the dual Brunn–Minkowski theory

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ABSTRACT

We introduce the dual isoperimetrix, which solves the isoperimetric problem in the dual Brunn–Minkowski theory. We then show how the dual isoperimetrix is related to the isoperimetrix from the Brunn–Minkowski theory.

1. Introduction and statement of main results

A *definition of volume* is a way of measuring volumes on finite-dimensional normed spaces, or, more generally, on Finsler manifolds. Roughly speaking, a definition of volume μ associates to each finite-dimensional normed space $(V, \|\cdot\|)$ a norm on $\Lambda^n V$, where $n = \dim V$. We refer to [3,28] and Section 2 for more information. The best

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known examples of definitions of volume are the *Busemann volume* μ^b, which equals the Hausdorff measure, the *Holmes–Thompson volume* μ^{ht}, which is related to symplectic geometry and *Gromov's mass** μ^{m*}, which thanks to its convexity properties is often used in geometric measure theory.

To each definition of volume μ may be associated a dual definition of volume μ^*. For instance, the dual of Busemann's volume is Holmes–Thompson volume and vice versa. The dual of Gromov's mass* is Gromov's mass, which however lacks good convexity properties and is used less often.

Given a definition of volume μ and a finite-dimensional normed vector space V, there is an induced $(n-1)$-density $\mu_{n-1} : \Lambda^{n-1}V \to \mathbb{R}$. Such a density may be integrated over $(n-1)$-dimensional submanifolds in V.

Given a compact convex set $K \subset V$, we let

$$A_\mu(K) := \int\limits_{\partial K} \mu_{n-1}$$

be the $(n-1)$-dimensional surface area of K.

If the boundary ∂K is smooth, each tangent space $T_p\partial K \subset V$ carries the induced norm and ∂K is a Finsler manifold. In the general case, one may make sense of the integral by using Alexandrov's theorem [1].

The definition of volume μ is called *convex*, if for compact convex bodies $K \subset L$, we have $A_\mu(K) \leqslant A_\mu(L)$. There are many equivalent ways of defining convexity of volume definitions, we refer to [3] for details. The above mentioned three examples are convex.

Given a convex definition of volume and an n-dimensional normed space V with unit ball B, there is a unique centrally symmetric compact convex body $\mathbb{I}_\mu B$ such that

$$A_\mu(K) = nV\big(K[n-1], \mathbb{I}_\mu B\big), \quad K \in \mathcal{K}(V).$$

Here V denotes the mixed volume and $\mathcal{K}(V)$ stands for the space of compact convex bodies in V. $\mathbb{I}_\mu B$ is called the isoperimetrix [3]. It was introduced in 1949 by Busemann [6] and has applications in crystallography [27,29] and in geometric measure theory [5,20,26].

As its name indicates, the isoperimetrix is related to isoperimetric problems. More precisely, Busemann [6] showed that among all compact convex bodies of a given, fixed volume, a homothet of the isoperimetrix has minimal surface area.

The isoperimetrices of the above mentioned examples of definitions of volume are related to important concepts from convex geometry. For Busemann's definition of volume, we have $\mathbb{I}_{\mu^b} B = \omega_{n-1}(IB)^\circ$, where $IB \subset V^*$ is the *intersection body* of B, $I^\circ B := (IB)^\circ \subset V$ its polar body, and ω_{n-1} the volume of the $(n-1)$-dimensional (Euclidean) unit ball. For the Holmes–Thompson volume, we have $\mathbb{I}_{\mu^{ht}} B = \frac{1}{\omega_{n-1}} \Pi(B^\circ)$, where Π denotes the *projection body*. It was shown recently by Ludwig [16] that the Holmes–Thompson surface area can be uniquely characterized by a valuation property. The isoperimetrix for Gromov's mass* is a dilate of the *wedge body* of B.