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Quasidiagonal representations of nilpotent groups

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ABSTRACT

We show that every unitary representation of a discrete solvable virtually nilpotent group G is quasidiagonal. Roughly speaking, this says that every unitary representation of G approximately decomposes as a direct sum of finite dimensional approximate representations. In operator algebraic terms we show that $C^*(G)$ is strongly quasidiagonal.

1. Introduction

Murray and von Neumann cite the study of unitary group representations as one of four key motivations for their development of operator algebra theory [20]. The last seventy-five years witnessed numerous intimate interactions between the theories, completely validating their motivation. The goal of this paper is to obtain yet another connection between representation theory and operator algebras. On one hand we use a natural approximation property of C*-algebras to obtain information about unitary representations of discrete nilpotent groups. On the other hand we employ classic results about nilpotent groups to produce some new examples of strongly quasidiagonal C*-algebras, and simple, nuclear quasidiagonal C*-algebras.

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A linear operator on a Hilbert space is called *quasidiagonal* if it is a compact perturbation of a direct sum of finite rank operators. One analogously defines quasidiagonality of a set of operators, and hence of a representation of a C*-algebra (see Definition 1.13). Interpreting quasidiagonality locally for a unitary group representation translates to declaring a unitary representation $\pi : G \to B(\mathcal{H})$ quasidiagonal if for every finite subset $\mathcal{F}$ of G and $\varepsilon > 0$, there are mutually orthogonal, finite rank projections $Q_n \in B(\mathcal{H})$ such that

$$\max_{t\in\mathcal{F}}\left\|\bigoplus_n Q_n\pi(t)Q_n - \pi(t)\right\| < \varepsilon.$$

Note that this implies that the function $t \mapsto Q_n\pi(t)Q_n$ is almost multiplicative on $\mathcal{F}$ and that $Q_n\pi(t)Q_n \in B(Q_n(\mathcal{H}))$ is almost unitary. In other words, π is quasidiagonal if it locally approximately decomposes as a direct sum of finite dimensional approximate unitary representations.

Rosenberg proved [10] that the left regular representation of a non-amenable discrete group is not quasidiagonal (see also [6] for a quantitative version of this theorem). It is a long-standing open question whether or not the left regular representation of every amenable group is quasidiagonal (see [6] for recent progress and the state of the art). Following Hadwin, we call a group *strongly quasidiagonal* if every unitary representation is quasidiagonal.

There are many examples in [6] of amenable groups whose left regular representation is quasidiagonal, while the group is not strongly quasidiagonal. The commonality between all of the examples is exponential growth; indeed it is precisely the growth conditions that lead to the non-quasidiagonal representations. On the other hand, it is fairly straightforward to see that every representation of an abelian group is quasidiagonal. Since nilpotent groups (Definition 1.14) posses a large degree of commutativity and have polynomial growth, it was natural to consider the problem of whether or not every nilpotent group is strongly quasidiagonal. Moreover there are representation theoretic simplifications present in nilpotent groups that suggest strong quasidiagonality. We recall the relevant facts.

Due to the fact that a discrete group is Type I if and only if it is virtually abelian [26] (a group is virtually “P” if it has a finite index “P” subgroup), to study representations of nilpotent groups one usually looks for replacements for the dual. There are two natural candidates for this replacement: The primitive ideal space of $C^*(G)$, and the space of characters on G.

An ideal of the group C*-algebra $C^*(G)$ is called *primitive* if it is the kernel of some irreducible representation of $C^*(G)$. The primitive ideal space of $C^*(G)$, denoted $\mathrm{Prim}(G)$, is equipped with the hull-kernel topology. In general, $\mathrm{Prim}(G)$ is topologically poorly behaved, but Moore and Rosenberg proved [18] that if G is nilpotent and finitely generated, then $\mathrm{Prim}(G)$ is T_1 (i.e., all of the singleton sets are closed). Poguntke later generalized their result with: