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η-Invariant and a problem of Bérard-Bergery on the existence of closed geodesics ☆

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A B S T R A C T

We use the η-invariant of Atiyah–Patodi–Singer to compute the Eells–Kuiper invariant for the Eells–Kuiper quaternionic projective plane. By combining with a known result of Bérard-Bergery, it shows that every Eells–Kuiper quaternionic projective plane carries a Riemannian metric such that all geodesics passing through a certain point are simply closed and of the same length.

1. Introduction

The η-invariant introduced by Atiyah, Patodi and Singer [2], as well as its various ramifications, has played important roles in many problems in geometry and topology. In this short paper, we use the η-invariant to compute the Eells–Kuiper invariant for the Eells–Kuiper quaternionic projective plane. By combining with a known result of

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Bérard-Bergery, it shows that every Eells–Kuiper quaternionic projective plane carries a Riemannian metric such that all geodesics passing through a certain point are simply closed and of the same length.

To be more precise, let p be a point in a closed manifold M. Let g be a Riemannian metric on M. The Riemannian structure (M, g) is called an SC^p Riemannian structure if all geodesics issued from p are simply closed (periodic) geodesics with the same length. We refer to the classic book [4] for a systematic account of the SC^p structures.

It is clear that there are SC^p Riemannian structures on the compact symmetric spaces of rank one (briefed in [4] as CROSS), namely the unit spheres, the real projective spaces, the complex projective spaces, the quaternionic projective spaces and the Cayley projective plane, endowed with the corresponding canonical metrics. Moreover, a fundamental result of Bott [5] states that any smooth manifold carrying an SC^p structure should have the same integral cohomology ring as that of a CROSS. On the other hand, there are manifolds verifying the above cohomological condition but not diffeomorphic to any CROSS. For typical examples, we mention the (exotic) homotopy spheres and the Eells–Kuiper (exotic) quaternionic projective planes.

In 1975, Bérard-Bergery [3] discovered an SC^p structure on an exotic sphere of dimension 10. He then raised the natural question: *is there any* (*exotic*) *Eells–Kuiper quaternionic projective plane carrying an* SC^p *structure*? The same question was also posed explicitly by Besse in the classic book [4, 0.15 on p. 4]. Moreover, it is pointed out in [4, p. 143] that a positive answer to the above question would also give a positive nontrivial example to the following open question: *whether a Blaschke manifold at a point*[1] *would carry an* SC^p *Riemannian structure*?

The purpose of this article is to provide a positive answer to the above two questions concerning the Eells–Kuiper quaternionic projective planes.

Before going on, we describe the Eells–Kuiper quaternionic projective planes as follows, starting with the standard construction of Milnor [15].

For any pair of integers (h, j), let $\xi_{h,j}$ be the S^3-bundle over S^4 determined by the characteristic map $f_{h,j} : S^3 \to SO(4)$ with $f_{h,j}(u)v = u^h v u^j$ for $u \in S^3$, $v \in \mathbf{R}^4$, where we identify $\mathbf{R}^4$ with the space of quaternions. It is shown in [15] that when $h + j = 1$, the total space of the above sphere bundle is homeomorphic to the unit sphere S^7. From now on, we denote by M_h this total space corresponding to $(h, j) = (h, 1 - h)$, and denote by N_h the associated disk bundle.

Remark 1.1. When $h = 0$ or 1, M_h is just the unit 7-sphere and the sphere bundle is just the Hopf fibration (corresponding to the left or right multiplications of the quaternions, respectively). On the other hand, M_2 is the exotic sphere generating the group $\Theta(7)$ (the set of the orientation preserving diffeomorphism classes of 7-dimensional oriented homotopy spheres), which is isomorphic to the cyclic group $\mathbf{Z}_{28}$.

[1] Cf. [4, 5.37 on p. 135] for a definition.