Advances in Mathematics 254 (2014) 69–78

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Localized energy equalities for the Navier–Stokes and the Euler equations

Dongho Chae

Department of Mathematics, Chung-Ang University, Seoul 156-756, Republic of Korea

ARTICLE INFO

Article history:
Received 4 October 2012
Accepted 25 December 2013
Available online 11 January 2014
Communicated by Charles Fefferman

MSC:
35Q30
35Q31
76D05

Keywords:
Navier–Stokes equations
Euler equations
Localized energy equality

ABSTRACT

Let (v,p) be a smooth solution pair of the velocity and the pressure for the Navier–Stokes (Euler) equations on $\mathbb{R}^N \times (0,T)$, $N \geqslant 3$. We set the Bernoulli function $Q = \frac{1}{2}|v|^2 + p$. Under suitable decay conditions at infinity for (v,p) we prove that for almost all $\alpha(t)$ and $\beta(t)$ defined on $(0,T)$ there holds

$$\int\limits_{\{\alpha(t)<Q(x,t)<\beta(t)\}} \left(\frac{1}{2}\frac{\partial}{\partial t}|v|^2 + \nu|\omega|^2\right) dx = \nu \int\limits_{\{Q(x,t)=\beta(t)\}} |\nabla Q|\, dS - \nu \int\limits_{\{Q(x,t)=\alpha(t)\}} |\nabla Q|\, dS,$$

where $\omega = \operatorname{curl} v$ is the vorticity. This shows that, in each region squeezed between two levels of the Bernoulli function, besides the energy dissipation due to the enstrophy, the energy flows into the region through the level hypersurface having the higher level, and the energy flows out of the region through the level hypersurface with the lower level. Passing $\alpha(t) \downarrow \inf_{x\in\mathbb{R}^N} Q(x,t)$ and $\beta(t) \uparrow \sup_{x\in\mathbb{R}^N} Q(x,t)$, we recover the well-known energy equality, $\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |v|^2 = -\nu \int_{\mathbb{R}^N} |\omega|^2\, dx$. A weaker version of the above equality under the weaker decay assumption of the solution at spatial infinity is also derived. The stationary version of the equality implies the previous Liouville type results on the Navier–Stokes equations.

E-mail address: dchae@cau.ac.kr.

0001-8708/$ – see front matter
http://dx.doi.org/10.1016/j.aim.2013.12.022

1. Introduction

We are concerned here with the incompressible Navier–Stokes (Euler) equations on $\mathbb{R}^N$, $N \geqslant 2$.

$$(NS, E) \qquad \begin{cases} v_t + (v \cdot \nabla)v + \nabla p = \nu \Delta v, \\ \operatorname{div} v = 0, \\ v(x,0) = v_0(x), \end{cases}$$

where $v = v(x,t) = (v_1(x,t), \dots, v_N(x,t))$ is the velocity, and $p = p(x,t)$ is the pressure. We assume the viscosity satisfies $\nu \geqslant 0$. In the case $\nu > 0$ the system (NS, E) becomes the Navier–Stokes equations, while for $\nu = 0$ the system (NS, E) is the Euler equations. By the system (NS, E) we represent both of the cases of the Navier–Stokes and the Euler equations. Below we denote the vorticity of the vector field v in $\mathbb{R}^N$ defined by $\omega = \{\partial_j v_k - \partial_k v_j\}_{j,k=1;\ j>k}^N$, the magnitude of which is given by

$$|\omega| = \sqrt{\frac{1}{2} \sum_{j,k=1}^N (\partial_j v_k - \partial_k v_j)^2}.$$

As is well-known in most of the fluid mechanics text books, the smooth solution v of the system (NS, E) with sufficiently fast decays at spatial infinity satisfies the energy equality,

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |v|^2 \, dx = -\nu \int_{\mathbb{R}^N} |\omega|^2 \, dx. \tag{1.1}$$

Our aim in this paper is to localize the domain of the integration in (1.1). The domains are characterized by their boundaries, which are the level hyper-surfaces of the Bernoulli function,

$$Q(x,t) := \frac{1}{2} \left| v(x,t) \right|^2 + p(x,t). \tag{1.2}$$

Our localized integral equalities refine the classical equality (1.1), in the sense that under suitable integrability conditions for the solutions and particular choice of the levels of the Bernoulli function we recover (1.1). The first theorem below concerns these localized energy equalities under milder conditions on the asymptotic behavior for the solution (v, p) at spatial infinity.

Theorem 1.1. *Let $N \geqslant 2$, and (v, p) be a smooth solution of (NS, E) on $\mathbb{R}^N \times (0, T)$. Let $Q(x,t)$ be defined as in (1.2). Suppose there exists $Q_0 = Q_0(t)$ such that*

$$\lim_{|x| \to \infty} Q(x,t) = Q_0(t) \tag{1.3}$$