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On the birational nature of lifting

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Abstract

Let X and Y be proper birational varieties, say with only rational double points over a perfect field k of positive characteristic. If X lifts to $W_n(k)$, is it true that Y has the same lifting property? This is true for smooth surfaces, but we show by example that this is false for smooth varieties in higher dimension, and for surfaces with canonical singularities. We also answer a stacky analogue of this question: given a canonical surface X with minimal resolution Y and stacky resolution $\mathcal{X}$, we characterize when liftability of Y is equivalent to that of $\mathcal{X}$.

The main input for our results is a study of how the deformation functor of a canonical surface singularity compares with the deformation functor of its minimal resolution. This extends work of Burns and Wahl to positive characteristic. As a byproduct, we show that Tjurina's vanishing result fails for every canonical surface singularity in every positive characteristic.

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1. Introduction

In 1961, Serre gave a surprising example of a smooth projective variety over a field of positive characteristic which admits no lifting to characteristic 0 [41]. The question of whether a variety

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admits such a lift is oftentimes subtle, and is intimately tied to pathological behavior in positive characteristic. In this paper, we explore the extent to which liftability is a birational invariant. Since many classification results and constructions in classical algebraic geometry yield singular varieties, and lifting is often easier to establish for these singular models (see, for example [28]), we will study varieties with mild singularities.

Question 1. Let X and Y be proper birational varieties of dimension d, say, with at worst rational double points over a perfect field k of positive characteristic. If Y lifts to $W_n(k)$, is it true that X also lifts to $W_n(k)$?

Note that this question has two main features: first, we put a bound on the singularities of X and Y; second, we ask for unramified lifts, namely lifts to $W_n(k)$ as opposed to extensions of $W_n(k)$. A bound on the singularities is certainly needed to make Question 1 meaningful. Indeed, every d-dimensional projective variety X is birational, via generic projection, to a hypersurface in $\mathbb{P}^{d+1}$. This hypersurface may have bad singularities (for example, non-normal), but it always lifts to $W(k)$. On the other hand, X may fail to lift.

Second, recall that there is an important distinction between unramified and ramified lifts of a variety. As is well-known, many fundamental theorems in characteristic 0 fail to hold in positive characteristic: global differential forms need not be closed [30] and Kodaira vanishing may fail to hold [33]. However, if X admits a lift to $W_2(k)$, by a result of Deligne and Illusie [16], these pathologies disappear. As examples of Lang show [24], even if a variety admits a lift to a ramified extension of $W(k)$ with the smallest possible ramification index, namely 2, this is not enough to ensure that global differential forms be closed. Hence, we restrict attention in Question 1 to the case of unramified lifts.

Question 1 is known to have a positive answer for smooth surfaces. In contrast, we prove the following result for higher dimensional varieties.

Theorem 1.1. *If $d \geqslant 3$, Question 1 has a negative answer, even if X and Y are smooth. In fact, if $d \geqslant 5$, there exist*

(a) *smooth blow-ups of $\mathbb{P}^d_k$ that do not lift to $W_2(k)$,*
(b) *smooth blow-ups of $\mathbb{P}^d_k$ that do not lift formally to any ramified extension of $W(k)$.*

Our specific counter-examples in dimensions 3 and 4 are given in Theorem 2.4. In Theorem 2.6 we give further examples of 3-folds with ordinary double points that lift to $W(k)$, but where small resolutions of singularities do not even lift to $W_2(k)$.

We next turn to the case of surfaces with singularities (see Theorem 3.4 for the counter-examples).

Theorem 1.2. *If $d = 2$, Question 1 again has a negative answer; however, if X has at worst rational singularities and Y is smooth, then Question 1 has a positive answer.*

Lastly, we explore a variant on Question 1 which constitutes the most subtle part of the paper. If X is a surface with canonical singularities, classically one studies the minimal resolution of singularities

$$f : Y \to X.$$