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The complexity of tropical matrix factorization

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ABSTRACT

The tropical arithmetic operations on $\mathbf{R}$ are defined by $a \oplus b = \min\{a, b\}$ and $a \otimes b = a + b$. Let A be a tropical matrix and k a positive integer, the problem of Tropical Matrix Factorization (TMF) asks whether there exist tropical matrices $B \in \mathbf{R}^{m \times k}$ and $C \in \mathbf{R}^{k \times n}$ satisfying $B \otimes C = A$. We show that the TMF problem is NP-hard for every $k \geqslant 7$ fixed in advance, thus resolving a problem proposed by Barvinok in 1993.

1. Introduction

The *tropical semiring* is the set $\mathbf{R}$ of real numbers equipped with the operations of tropical addition and tropical multiplication, which are defined by $a \oplus b = \min\{a, b\}$, $a \otimes b = a + b$. The tropical semiring is essentially the same structure as the *max-plus algebra*, which is the set $\mathbf{R}$ with the operations of maximum and sum, and is being studied since the 1960s, when the applications in the optimization theory have been found [31]. The tropical arithmetic operations on $\mathbf{R}$, which allow us to formulate a number of important non-linear problems in a linear-like way, arise indeed in a variety of topics in

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pure and applied mathematics. The study of tropical mathematics has applications in operations research [12], discrete event systems [2], automata theory [30], optimal control [27,28], algebraic geometry [15,16], and others; we refer to [20] for a detailed survey of applications. A considerable number of important problems in tropical mathematics has a linear-algebraic nature. For instance, the concepts of eigenvalue and eigenvector, the theory of linear systems, and the algorithms for computing rank functions are useful for different applications [1,15,23,20]. Some applications also give rise to studying the multiplicative structure of tropical matrices [29], and in this context, the Burnside-type problems are important [19,30]. Another interesting problem is to study the subgroup structure of the semigroup of tropical matrices under multiplication [25,26].

In our paper, we consider the problem of matrix factorization, which is also related to the concept of factor rank of matrices over semirings [9]. The study of factor rank dates back to the 1980s [8], and has now numerous applications in different contexts of mathematics. Being considered on the semiring of nonnegative matrices, the factor rank is known as nonnegative rank and has applications in quantum mechanics, statistics, demography, and others [11]. The factor rank of matrices over the binary Boolean semiring is also called Boolean rank and has applications in combinatorics and graph theory [7,21]. Finally, for matrices over a field, the factor rank coincides with the classical rank function.

In the context of matrices over the tropical semiring, the factor rank is also known as *combinatorial rank* [4] and *Barvinok rank* [15], and the study of this notion has arisen from combinatorial optimization [3]. The factor rank appears in the formulation of a number of problems in optimization, for instance, in the Traveling Salesman problem with warehouses [6]. Also, the notion of factor rank is of interest in the study of tropical geometry [14,15], where the factor rank can be thought of as the minimum number of points whose tropical convex hull contains the columns of a matrix. Let us define the factor rank function for tropical matrices, assuming that multiplication of tropical matrices is understood as ordinary matrix multiplication with $+$ and $\cdot$ replaced by the tropical operations $\oplus$ and $\otimes$.

Definition 1.1. The *factor rank* of a tropical matrix $A \in \mathbf{R}^{m\times n}$ is the smallest integer k for which there exist tropical matrices $B \in \mathbf{R}^{m\times k}$ and $C \in \mathbf{R}^{k\times n}$ satisfying $B \otimes C = A$.

The most straightforward way of computing the factor rank is based on the quantifier elimination algorithm for the theory of reals with addition and order [17]. Indeed, Definition 1.1 allows us to define the set of all m-by-n matrices with factor rank k by a first-order order formula. We then employ the decision procedure based on the quantifier elimination algorithm provided in [17] to check whether a given m-by-n matrix indeed has factor rank k.

However, the computational complexity of quantifier elimination makes the algorithm mentioned unacceptable for practical use. Another algorithm for computing the factor rank is given by Develin in the paper [14], where he develops the theory of tropical secant