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Models for singularity categories

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ABSTRACT

In this article we construct various models for singularity categories of modules over differential graded rings. The main technique is the connection between abelian model structures, cotorsion pairs and deconstructible classes, and our constructions are based on more general results about localization and transfer of abelian model structures. We indicate how recollements of triangulated categories can be obtained model categorically, discussing in detail Krause's recollement $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(R)) \to \mathrm{K}(\mathrm{Inj}(R)) \to \mathrm{D}(R)$. In the special case of curved mixed $\mathbb{Z}$-graded complexes, we show that one of our singular models is Quillen equivalent to Positselski's contraderived model for the homotopy category of matrix factorizations.

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0. Introduction

Let R be a Noetherian ring and $\mathrm{D}_{\mathrm{sg}}(R) = \mathrm{D}^b(R\text{-mod})/\operatorname{Perf}(R)$ its singularity category. We ask if it is possible to realize $\mathrm{D}_{\mathrm{sg}}(R)$ as the homotopy category of a stable model category attached to R. Firstly, the singularity category is essentially small, whereas the homotopy category of a model category in the sense of [12] always has arbitrary small coproducts [12, Example 1.3.11]. This forces us to think first about how to define a "large" singularity category for R (admitting arbitrary small coproducts) in which $\mathrm{D}_{\mathrm{sg}}(R)$ naturally embeds. Secondly, if this is done, we can try to find a model for this large singularity category.

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Given a locally Noetherian Grothendieck category $\mathscr{A}$ with compactly generated derived category $\mathrm{D}(\mathscr{A})$, Krause [16] proved that the singularity category $\mathrm{D}^b(\mathrm{Noeth}(\mathscr{A}))/\mathrm{D}(\mathscr{A})^{\mathrm{c}}$ of $\mathscr{A}$ (the Verdier quotient of the bounded derived category of Noetherian objects of $\mathscr{A}$ by the subcategory of compact objects of $\mathrm{D}(\mathscr{A})$) is up to direct summands equivalent to the subcategory of compact objects in the homotopy category $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathscr{A}))$ of acyclic complexes of injectives, and that there is even a recollement $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathscr{A})) \leftrightarrows \mathrm{K}(\mathrm{Inj}(\mathscr{A})) \leftrightarrows \mathrm{D}(\mathscr{A})$. This suggests firstly that we should attempt to construct a model for $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathscr{A}))$ and secondly that such a model might be obtained by localizing a suitable model for $\mathrm{K}(\mathrm{Inj}(\mathscr{A}))$ with respect to $\mathrm{D}(\mathscr{A})$, whatever this should mean precisely.

If $\mathscr{A} = R$-Mod for a Noetherian ring R, Positselski [20, Theorem 3.7] showed that $\mathrm{K}(\mathrm{Inj}(\mathscr{A}))$ is equivalent to what he calls the *coderived category* $\mathrm{D}^{\mathrm{co}}(R)$ of R, defined as the Verdier quotient $\mathrm{K}(R)/\operatorname{Acyc}^{\mathrm{co}}(R)$, where $\operatorname{Acyc}^{\mathrm{co}}(R)$ is the localizing subcategory of $\mathrm{K}(R)$ generated by the total complexes of short exact sequences of complexes of R-modules; objects of $\operatorname{Acyc}^{\mathrm{co}}(R)$ are called *coacyclic complexes*. In particular, Krause's "large" singularity category $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$ is equivalent to a Verdier quotient $\mathrm{D}^{\mathrm{co}}(R)/\mathrm{D}(R)$.

All in all, the last paragraphs suggest that a model for the singularity category could be obtained by lifting the quotient $\mathrm{D}^{\mathrm{co}}(R)/\mathrm{D}(R)$ to the world of model categories. For $\mathrm{D}(R)$ there are the well-known projective and injective models, and for $\mathrm{D}^{\mathrm{co}}(R)$ a model has been constructed by Positselski [20]. Moreover, these models are *abelian*, i.e. they are compatible with the abelian structure of $\mathrm{Ch}(R\text{-Mod})$ in the sense of [13, Definition 2.1]. By [13, Theorem 2.2] an abelian model structure is completely determined by the classes $\mathcal{C}$, $\mathcal{W}$, $\mathcal{F}$ of cofibrant, weakly trivial and fibrant objects, respectively, and the triples $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ arising in this way are precisely those for which $\mathcal{W}$ is thick and both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs. For example, in the injective model $\mathcal{M}^{\mathrm{inj}}(R)$ for $\mathrm{D}(R)$, everything is cofibrant, the weakly trivial objects $\mathcal{W}^{\mathrm{inj}}$ are the acyclic complexes and the fibrant objects $\mathcal{F}^{\mathrm{inj}}$ are the dg-injectives. In Positselski's coderived model $\mathcal{M}^{\mathrm{co}}(R)$ for $\mathrm{D}^{\mathrm{co}}(R)$, again everything is cofibrant, but the weakly trivial objects $\mathcal{W}^{\mathrm{co}}$ are the coacyclic complexes (see Proposition 1.3.6) and the fibrant objects $\mathcal{F}^{\mathrm{co}}$ are the componentwise injective complexes of R-modules. In particular, we see that both model structures are *injective* in the sense that everything is cofibrant, and that $\mathcal{W}^{\mathrm{co}}(R) \subset \mathcal{W}^{\mathrm{inj}}(R)$ and $\mathcal{F}^{\mathrm{inj}}(R) \subset \mathcal{F}^{\mathrm{co}}(R)$.

In order to construct the desired localization, we show (Proposition 1.4.2) that given an abelian category $\mathscr{A}$ with two injective abelian model structures $\mathcal{M}_i = (\mathscr{A}, \mathcal{W}_i, \mathcal{F}_i)$, $i = 1, 2$, satisfying $\mathcal{F}_2 \subset \mathcal{F}_1$ (hence $\mathcal{W}_1 \subset \mathcal{W}_2$), there is another new abelian model structure $\mathcal{M}_1/\mathcal{M}_2$ on $\mathscr{A}$ with $\mathcal{C} = \mathcal{W}_2$ and $\mathcal{F} = \mathcal{F}_1$ (the class $\mathcal{W}$ of weakly trivials is determined by this and described explicitly in the proposition), called the *right localization* of $\mathcal{M}_1$ with respect to $\mathcal{M}_2$. Moreover, we show (Proposition 1.5.3) that $\mathcal{M}_1/\mathcal{M}_2$ is a right Bousfield localization of $\mathcal{M}_1$ with respect to $\{0 \to X \mid X \in \mathcal{F}_2\}$ in the sense of [11, Definition 3.3.1(2)], and that on the level of homotopy categories we get a colocalization sequence [16, Definition 3.1] of triangulated categories $\mathrm{Ho}(\mathcal{M}_2) \to \mathrm{Ho}(\mathcal{M}_1) \to \mathrm{Ho}(\mathcal{M}_1/\mathcal{M}_2)$.

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